

Application of abelian holonomy formalism to the elementary theory of numbers

YASUHIRO ABE

Cereja Technology Co., Ltd.
3-1 Tsutaya-Bldg. 5F, Shimomiyabi-cho
Shinjuku-ku, Tokyo 162-0822, Japan
 abe@cereja.co.jp

Abstract

We consider an abelian holonomy operator in two-dimensional conformal field theory with zero-mode contributions. The analysis is made possible by use of a geometric-quantization scheme for abelian Chern-Simons theory on $S^1 \times S^1 \times \mathbf{R}$. We find that a purely zero-mode part of the holonomy operator can be expressed in terms of Riemann's zeta function. We also show that a generalization of linking numbers can be obtained in terms of the vacuum expectation values of the zero-mode holonomy operators. Inspired by mathematical analogies between linking numbers and Legendre symbols, we then apply these results to a space of $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ where p is an odd prime number. This enables us to calculate “scattering amplitudes” of identical odd primes in the holonomy formalism. In this framework, the Riemann hypothesis can be interpreted by means of a physically obvious fact, *i.e.*, there is no notion of “scattering” for a single-particle system. Abelian gauge theories described by the zero-mode holonomy operators will be useful for studies on quantum aspects of topology and number theory.

1 Introduction

The concept of holonomy in conformal field theory or, more precisely, holonomy of what is called the Knizhnik-Zamolodchikov (KZ) connection in mathematics has been developed in search of a geometric understanding of the conformal field theory [1]. This geometric approach has an advantage in obtaining topological invariants. Particularly, it is known that the holonomy is useful in construction of the so-called Witten invariant which is originally formulated as a partition function of three-dimensional Chern-Simons theory [2]. The holonomy in conformal field theory is therefore closely related to the knot theory. In fact, the holonomy can be defined as a linear representation of a braid group. Since knots, and links in general, can be generated by the braid group, the holonomy is indeed an indispensable element in application of (conformal) field theory to topology.

Recently, use of such a holonomy operator in a form that is more familiar to physicists is proposed as a non-perturbative formulation of non-abelian gauge theories [3]. In this approach, which we call holonomy formalism, tree-level scattering amplitudes of gluons are shown to be generated by a holonomy operator that is defined in a \mathbf{CP}^1 fiber of twistor space. Notice that twistor space \mathbf{CP}^3 can be regarded as a \mathbf{CP}^1 -bundle over compact spacetime S^4 ; thus twistor space (or its supersymmetric version) is naturally required in order to incorporate four-dimensional spacetime information. An essential structure of the holonomy formalism is, however, encoded in the holonomy operator of the \mathbf{CP}^1 fiber on which a two-dimensional conformal field theory is defined. This formulation is developed along the line of the so-called spinor-momenta or spinor-helicity formalism. (For details of this formalism, see [3] and references therein.) In order to further understand the validity and usefulness of the holonomy formalism and its essentials, it is therefore natural to investigate an abelian holonomy operator in conformal field theory without any spacetime or twistor-space information. This is a main motivation of the present paper.

The abelian holonomy operator that we discuss here serves as a topological “skeleton” of abelian gauge theories, including, in particular, Maxwell’s theory of electromagnetism. Thus topological invariants that can be generated by the abelian holonomy should include Gauss’s linking number. It is well-known that the linking numbers can be expressed in terms of a partition function of abelian Chern-Simons theory (see, for example, [4]-[6]). The linking numbers can also be expressed as degrees of mapping, or winding

numbers, of a map $\phi : S^1 \times S^1 \longrightarrow S^2$. (For the notion of winding number, see, *e.g.*, [7].) In fact, the linking numbers naturally arise from an abelian Chern-Simons theory on $S^1 \times S^1 \times \mathbf{R}$ which can be interpreted as a Wess-Zumino-Witten (WZW) model on torus. As we review in section 3, this can neatly be shown by use of geometric quantization for the toric abelian Chern-Simons theory [8, 9, 10]. We shall apply this geometric scheme to the abelian holonomy formalism such that generalization of linking numbers can be obtained in terms of the abelian holonomy operator. Generalization and application of linking numbers in different physical contexts can also be found in [11]-[14].

In the present paper, we try to convey a novel perspective on the linking numbers by applying the abelian holonomy formalism to the elementary theory of numbers. This is motivated by the following mathematical results. Interestingly, the concept of linking number can be applied to odd primes and, by use of Galois groups, one can define a linking number (mod 2) of p and q , denoted by $lk(p, q)$, where p and q are two distinct odd primes. This is shown by Morishita [15]. We do not discuss details of $lk(p, q)$ here but the upshot of Morishita's result is that $lk(p, q)$ is related to the Legendre symbol by

$$(-1)^{lk(p, q)} = \left(\frac{q^*}{p} \right) \quad (1.1)$$

where $q^* = (-1)^{\frac{q-1}{2}} q$. The Legendre symbol $\left(\frac{q^*}{p} \right)$ is defined by

$$\left(\frac{q^*}{p} \right) = \lambda_p(q^*) = \begin{cases} +1 & \text{if } q^* \text{ is a quadratic residue modulo } p; \\ -1 & \text{otherwise.} \end{cases} \quad (1.2)$$

That q^* is a quadratic residue modulo p means that there exists an integer n such that $n^2 \equiv q^* \pmod{p}$. In the elementary theory of numbers, the Legendre symbol appears in the Gauss sum

$$\hat{\lambda}_p = \sum_{x=1}^{p-1} \lambda_p(x) e^{i \frac{2\pi}{p} x}. \quad (1.3)$$

Notice that the sum is taken over the elements of $\mathbf{F}_p^\times = \mathbf{F}_p - \{0\}$ where $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. Since the Gauss sum takes a value of complex number, the Legendre symbol gives a map

$$\lambda_p : \mathbf{F}_p^\times \rightarrow \mathbf{C}. \quad (1.4)$$

This suggests that we can interpret $\widehat{\lambda}_p$ as a Fourier transform of $\lambda_p(x)$ in a space of mod p . (For description of the Gauss sum as a Fourier transform, see, *e.g.*, [16].)

At first glance, the relations (1.1)-(1.4) seem irrelevant to the algebraic construction of holonomies. However, as mentioned earlier, the holonomy formalism is closely related to knot theory and linking numbers can be generated by an abelian holonomy operator. Thus, the facts that the linking number is essentially equivalent to the Legendre symbol and that the Gauss sum can be interpreted as a Fourier transform of the Legendre symbol in a space of finite field \mathbf{F}_p^\times are quite suggestive. Namely, application of the holonomy formalism to elementary number theory will be useful to obtain something analogous to scattering amplitudes of prime numbers. In other words, it is presumably possible to express a generating function for such “scattering amplitudes” of primes in terms of the abelian holonomy operator suitably defined in \mathbf{F}_p^\times . A main goal of the present paper is to give an explicit realization of such a generating function. Roughly speaking, this means that our main goal is to obtain a field theoretic description of the idea that there is a correspondence between knots and primes, *i.e.*,

$$(\text{integer}) = \prod_i (\text{prime})_i \longleftrightarrow \prod_i (\text{knot})_i = (\text{link}), \quad (1.5)$$

in the context of holonomy formalism. For mathematical studies on the relation between knots and primes, one may refer to [17].

Another intriguing aspect of the abelian holonomy formalism which we would like to deliver in the present paper is that, if we include zero-mode contributions and consider only the zero-mode part of the abelian holonomy operator, then the zero-mode holonomy operator can be expressed in terms of Riemann’s zeta function. It is well-known that Riemann’s zeta function can be represented by an iterated integral [1]. In section 4, we shall show that the zero-mode part of the abelian holonomy operator contains essentially the same iterated-integral representation as Riemann’s zeta function. The fact that the abelian holonomy operator is represented by an iterated integral is not surprising if we remember that the WZW model can be defined by an iterative integral [18]. (For the use of an iterative integral in physics, see also [19].)

Riemann’s zeta function is intimately related to prime numbers by the formula of Euler’s product. In applying the zero-mode holonomy operator to a space of \mathbf{F}_p^\times , we shall make a detailed argument that one can interpret the Gauss sum as an operator which is relevant to creation of odd primes. Along

with the discussion on the generating function for “scattering amplitudes” of primes, we then further investigate the zero-mode holonomy operator in relation to quantum realization of Riemann’s zeta function. We propose that clarification of this relation will lead to a new, if speculative, physical interpretation of the Riemann hypothesis. (For recent studies on related topics, see, *e.g.*, [20]-[22].)

The paper is organized as follows. In the next section, we briefly review the holonomy formalism, following [3], and present an explicit definition of the abelian holonomy operator. In section 3, we review the geometric-quantization scheme for the toric $U(1)$ Chern-Simons theory, following [10]. We shall show that linking numbers naturally arise from a holomorphic wavefunction of the toric $U(1)$ Chern-Simons theory. In section 4, utilizing the results in the previous sections, we construct and calculate an abelian holonomy operator with zero-mode contributions. We show that generalization of linking numbers can be obtained from such a holonomy operator. We also show that the zero-mode part of the abelian holonomy operator can be expressed in terms of Riemann’s zeta function by use of an iterated-integral representation. In section 5, we apply the zero-mode holonomy operator to a space of \mathbf{F}_p^\times and discuss how one can obtain a generating function for “scattering amplitudes” of prime numbers. We shall also propose a new interpretation of the Riemann hypothesis in the context of abelian holonomy formalism. Lastly, we shall present some concluding remarks.

2 Review of abelian holonomy operators

In this section we review essentials of the holonomy formalism, following the construction introduced in [3]. The holonomy formalism is originally developed in explaining multi-gluon amplitudes in helicity-based calculations. Basic physical operators are then given by creation operators of gluons with helicity \pm , which can be identified as ladder operators of the $SL(2, \mathbf{C})$ algebra; these are denoted by $a_i^{(\pm)}$ in (2.1). An abelian version of the formalism is thus obtained simply by replacing a role of gluons with that of photons. As mentioned earlier, in this paper we shall focus on holonomies of two-dimensional conformal field theories, rather than dealing with twistor space to construct four-dimensional theories. Therefore, strictly speaking, the notion of helicity and even that of 4-momentum are not appropriate. The use of $SL(2, \mathbf{C})$ algebra is however persistent in either case. Thus we can still use $a_i^{(\pm)}$ as physical operators in two dimensions. Here the sign \pm of $a_i^{(\pm)}$ no

longer corresponds to helicity but some analog of it.

Braid groups, KZ equation and comprehensive gauge fields

Bearing in mind the above notes on helicity, we now consider a multi-photon system in the holonomy formalism. The Hilbert space of the system is given by $V^{\otimes n} = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ where V_i ($i = 1, 2, \dots, n$) denotes a Fock space that creation operators of the i -th particle with helicity \pm act on. Such physical operators $a_i^{(\pm)}$ are given by ladder operators that form a part of the $SL(2, \mathbf{C})$ algebra. The algebra can be expressed as

$$[a_i^{(+)}, a_j^{(-)}] = 2a_i^{(0)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(+)}] = a_i^{(+)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(-)}] = -a_i^{(-)} \delta_{ij} \quad (2.1)$$

where Kronecker's deltas show that the non-zero commutators are obtained only when $i = j$. The remaining of commutators, those expressed otherwise, all vanish.

The physical configuration space of n photons is given by $\mathcal{C} = \mathbf{C}^n / \mathcal{S}_n$, where \mathcal{S}_n is the rank- n symmetric group. The \mathcal{S}_n arises from the fact that photons are bosons. The complex number \mathbf{C} corresponds to a local coordinate of \mathbf{CP}^1 on which, in the context of spinor-momenta formalism, a spinor momentum of the i -th photon is defined. In this paper we do not strictly follow the spinor-momenta formalism. Thus it is not appropriate to use a momentum-representation of $a_i^{(\pm)}$. However, for a purpose of obtaining abelian holonomy operators of conformal field theory, we can still label these operators in terms of complex coordinates of \mathbf{CP}^1 because it is this \mathbf{CP}^1 where a conformal field theory is defined. In the next section, we shall deform the topology of $\mathbf{CP}^1 = S^2$ to that of torus $T^2 = S^1 \times S^1$ and shall consider zero-mode contributions to the abelian holonomies.

It is well-known that the fundamental homotopy group of $\mathcal{C} = \mathbf{C}^n / \mathcal{S}_n$ is given by the braid group, $\Pi_1(\mathcal{C}) = \mathcal{B}_n$. The braid group \mathcal{B}_n has generators, b_1, b_2, \dots, b_{n-1} , and they satisfy the following relations:

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} & \text{if } |i - j| = 1, \\ b_i b_j &= b_j b_i & \text{if } |i - j| > 1 \end{aligned} \quad (2.2)$$

where we identify b_n with b_1 . To be mathematically rigorous, the \mathbf{C} of $\mathcal{C} = \mathbf{C}^n / \mathcal{S}_n$ should be replaced by \mathbf{CP}^1 which is represented by the local coordinate z . Since \mathbf{C} can be obtained from \mathbf{CP}^1 by excluding points at infinity, the replacement can be done with ease. At the level of braid generators, this can be carried out by imposing the following relation [23]

$$(b_1 b_2 \cdots b_{n-2} b_{n-1})(b_{n-1} b_{n-2} \cdots b_2 b_1) = 1. \quad (2.3)$$

The braid group that satisfies this condition on top of (2.2) is called a *sphere* braid group $\mathcal{B}_n(\mathbf{CP}^1)$, while the previous one is called a *pure* braid group $\mathcal{B}_n(\mathbf{C}) = \mathcal{B}_n$. Thus, bearing in mind this subsidiary condition, we can identify $\mathcal{C} = \mathbf{C}^n/\mathcal{S}_n$ as the physical configuration space of interest.

Now, mathematically, a linear representation of a braid group is equivalent to a monodromy representation of the Knizhnik-Zamolodchikov (KZ) equation. The KZ equation is an equation that a function on \mathcal{C} satisfies in general. We can denote such a function as $\Psi(z_1, z_2, \dots, z_n)$, where z_i ($i = 1, 2, \dots, n$) represents the local complex coordinate of \mathbf{CP}^1 . The KZ equation is then expressed as

$$\frac{\partial \Psi}{\partial z_i} = \frac{1}{\kappa} \sum_{j(j \neq i)} \frac{\Omega_{ij} \Psi}{z_i - z_j} \quad (2.4)$$

where κ is a non-zero constant called the KZ parameter. We now introduce logarithmic differential one-forms

$$\omega_{ij} = d \log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}. \quad (2.5)$$

Notice that these satisfy the identity

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0 \quad (2.6)$$

where the indices are ordered as $i < j < k$. The quantity Ω_{ij} in the KZ equation is a bialgebraic operator. In terms of the operators of $SL(2, \mathbf{C})$ algebra in (2.1), this can be defined as

$$\Omega_{ij} = a_i^{(+)} \otimes a_j^{(-)} + a_i^{(-)} \otimes a_j^{(+)} + 2a_i^{(0)} \otimes a_j^{(0)}. \quad (2.7)$$

In the case of $i = j$, this becomes the quadratic Casimir of $SL(2, \mathbf{C})$ algebra which acts on the i -th Fock space V_i . Action of Ω_{ij} on $V^{\otimes n} = V_1 \otimes V_2 \otimes \dots \otimes V_n$ can be written as

$$\sum_{\mu} 1 \otimes \dots \otimes 1 \otimes \rho_i(I_{\mu}) \otimes 1 \otimes \dots \otimes 1 \otimes \rho_j(I_{\mu}) \otimes 1 \otimes \dots \otimes 1 \quad (2.8)$$

where I_{μ} ($\mu = 0, 1, 2$) are elements of the $SL(2, \mathbf{C})$ algebra and ρ denotes its representation. Introducing the one-form

$$\Omega = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} \Omega_{ij} \omega_{ij}, \quad (2.9)$$

we can then rewrite the KZ equation (2.4) as a differential equation

$$D\Psi = (d - \Omega)\Psi = 0 \quad (2.10)$$

where $D = d - \Omega$ can be regarded as a covariant exterior derivative.

From the explicit form of (2.7), one can show

$$[\Omega_{ij}, \Omega_{kl}] = 0 \quad (i, j, k, l \text{ are distinct}), \quad (2.11)$$

$$[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0 \quad (i, j, k \text{ are distinct}). \quad (2.12)$$

In mathematical literature, these are called the infinitesimal braid relations. Remarkably, by use of these relations along with (2.6), the flatness of Ω , *i.e.*, $d\Omega - \Omega \wedge \Omega = 0$, can be shown. (For a proof of this, see [1, 3].) Therefore, it is possible to define a holonomy of Ω , which gives a general linear representation of a braid group on the Hilbert space $V^{\otimes n}$. This is the monodromy representation of the KZ equation. The Hilbert space $V^{\otimes n}$ can then be identified as the space of conformal blocks for the KZ equation.

We now consider applications of these mathematical results to the multi-photon system. As mentioned before, the physical operators of photons are given by $a_i^{(\pm)}$. The operator Ω_{ij} may not be appropriate to describe photons since its action on the Hilbert space is represented by (2.8), which involves $a_i^{(0)}$. We need to modify Ω_{ij} 's such that the operators $a_i^{(0)}$ are treated somewhat unphysically. We then introduce a following ‘‘comprehensive’’ gauge one-form

$$A = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} A_{ij} \omega_{ij} \quad (2.13)$$

where A_{ij} is defined as a bialgebraic operator

$$A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}. \quad (2.14)$$

Notice that, from the explicit form of A_{ij} , we can also show that the bialgebraic quantity A_{ij} satisfy the relations (2.11) and (2.12). (For details of this fact, see [3].) These relations are the only conditions for the flatness or integrability of A . Thus, as in the case of Ω , we can also obtain the expression

$$DA = dA - A \wedge A = -A \wedge A = 0 \quad (2.15)$$

where D is now a covariant exterior derivative $D = d - A$. This relation guarantees the existence of holonomies for the comprehensive gauge field A .

Although the bialgebraic structures of Ω and A are different, the constituents of these remain the same, *i.e.*, they are given by $a_i^{(0)}$ and $a_i^{(\pm)}$. Thus, we can use the same Hilbert space $V^{\otimes n}$ and physical configuration \mathcal{C} for both Ω and A . The KZ equation of A is then given by $D\Psi = (d - A)\Psi = 0$, where Ψ is a function of a set of complex variables (z_1, z_2, \dots, z_n) . This suggests that the inverse of the KZ parameter κ can be interpreted as a coupling constant of the gauge theory.

Abelian holonomy operators

A holonomy of A can be given by a general solution to the KZ equation $D\Psi = (d - A)\Psi = 0$. The construction is therefore similar to that of Wilson loop operators. In the present formalism, rank- n differential manifolds are physically relevant for the construction. Thus, we need differential n -forms in terms of A in order to define an appropriate holonomy operator. Further, an analog of Wilson loop should be defined on \mathcal{C} . These requirements lead to the following definition of the holonomy operator.

$$\Theta_{R,\gamma}(z) = \text{Tr}_{R,\gamma} \text{P exp} \left[\sum_{r \geq 2} \oint_{\gamma} \underbrace{A \wedge A \wedge \dots \wedge A}_r \right] \quad (2.16)$$

where γ represents a closed path on \mathcal{C} along which the integral is evaluated and R denotes the representation of a gauge group. For the abelian group $U(1)$, the representation can be trivially determined. In the following, we explain the meanings of the symbol P and the trace $\text{Tr}_{R,\gamma}$, respectively.

The symbol P denotes a “path ordering” of the numbering indices i, j . The meaning of this symbol can be understood as follows. The exponent in (2.16) can be expanded as

$$\sum_{r \geq 2} \oint_{\gamma} \underbrace{A \wedge \dots \wedge A}_r = \sum_{r \geq 2} \oint_{\gamma} \sum_{(i < j)} A_{i_1 j_1} A_{i_2 j_2} \dots A_{i_r j_r} \bigwedge_{k=1}^r \omega_{i_k j_k} \quad (2.17)$$

where $(i < j)$ means that the set of indices $(i_1, j_1, \dots, i_r, j_r)$ are ordered such that $1 \leq i_1 < j_1 \leq r, \dots, 1 \leq i_r < j_r \leq r$. Notice that the coupling constant $1/\kappa$ is absorbed in A ; we shall use this convention unless otherwise mentioned in the following. Notice also that, if we have $r = 1$, we can define (2.17), or the exponent of (2.16), as zero. For $r = 1$, we cannot define either A defined in (2.13) or ω_{ij} in (2.5) but it is natural to consider ω_{ij} vanishing since in this case the quantity $(z_i - z_j)$ can be treated as a fixed variable. (We shall come back to this definition later in the end of section 5.) The symbol

P means that the numbering indices are further received ordering conditions $1 \leq i_1 < i_2 < \cdots < i_r \leq r$ and $2 \leq j_1 < j_2 < \cdots < j_r \leq r+1$ where $r+1$ is to be identified with 1. This automatically leads to the following expression

$$P \sum_{r \geq 2} \oint_{\gamma} \underbrace{A \wedge \cdots \wedge A}_r = \sum_{r \geq 2} \oint_{\gamma} A_{12} A_{23} \cdots A_{r1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{r1} \quad (2.18)$$

which shows that the basis of the holonomy operators (2.16) is given by $\omega_{i i+1}$ under the “path ordering” operation for the numbering indices.

Now the commutator $[A_{12}, A_{23}]$ can be calculated as

$$\begin{aligned} [A_{12}, A_{23}] &= a_1^{(+)} \otimes a_2^{(+)} \otimes a_3^{(0)} - a_1^{(+)} \otimes a_2^{(-)} \otimes a_3^{(0)} \\ &\quad + a_1^{(-)} \otimes a_2^{(+)} \otimes a_3^{(0)} - a_1^{(-)} \otimes a_2^{(-)} \otimes a_3^{(0)}. \end{aligned} \quad (2.19)$$

Equation (2.18) is then written as

$$\begin{aligned} P \sum_{r \geq 2} \oint_{\gamma} \underbrace{A \wedge \cdots \wedge A}_r &= \sum_{r \geq 2} \oint_{\gamma} A_{12} A_{23} \cdots A_{r1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{r1} \\ &= \sum_{r \geq 2} \frac{1}{2^{r+1}} \sum_{(h_1, h_2, \dots, h_r)} (-1)^{h_1 + h_2 + \cdots + h_r} \\ &\quad \times a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \oint_{\gamma} \omega_{12} \wedge \cdots \wedge \omega_{r1} \end{aligned} \quad (2.20)$$

where h_i denotes $h_i = \pm = \pm 1$ ($i = 1, 2, \dots, r$). In the above expression, we define $a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \otimes a_1^{(0)}$ as

$$\begin{aligned} a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \otimes a_1^{(0)} &\equiv \frac{1}{2} [a_1^{(0)}, a_1^{(\pm)}] \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \\ &= \pm \frac{1}{2} a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \end{aligned} \quad (2.21)$$

where we implicitly use an antisymmetric property for the indices $(1, 2, \dots, r)$ as indicated in (2.17) or (2.18).

The trace $\text{Tr}_{R, \gamma}$ in the definition (2.16) means traces over the generators of a $U(1)$ group and a braid group. The former trace is trivial and the latter, which is called a braid trace, can be realized by a sum over permutations of the numbering indices. Thus the trace $\text{Tr}_{R, \gamma}$ over the exponent of (2.16) can

be expressed as

$$\begin{aligned} & \text{Tr}_\gamma \text{P} \sum_{r \geq 2}^\infty \oint_\gamma \underbrace{A \wedge \cdots \wedge A}_r \\ &= \sum_{r \geq 2} \sum_{\sigma \in \mathcal{S}_{r-1}} \oint_\gamma A_{1\sigma_2} A_{\sigma_2\sigma_3} \cdots A_{\sigma_{r-1}} \omega_{1\sigma_2} \wedge \omega_{\sigma_2\sigma_3} \wedge \cdots \wedge \omega_{\sigma_{r-1}} \quad (2.22) \end{aligned}$$

where the summation of \mathcal{S}_{r-1} is taken over the permutations of the elements $\{2, 3, \dots, r\}$, with the permutations labeled by $\sigma = \begin{pmatrix} 2 & 3 & \cdots & r \\ \sigma_2 & \sigma_3 & \cdots & \sigma_r \end{pmatrix}$.

From the expression (2.20), we find that, with a suitable normalization for $\oint_\gamma \omega_{12} \wedge \cdots \wedge \omega_{r1}$, the holonomy operator can be used as a generating function for all physical states of photons on the Hilbert space $V^{\otimes n}$. The holonomy operator (2.16) is therefore a very universal operator. As a summary of this section, we recapitulate the definition of the abelian holonomy operator $\Theta_{R,\gamma}(z)$ below.

$$\Theta_{R,\gamma}(z) = \text{Tr}_{R,\gamma} \text{P} \exp \left[\sum_{r \geq 2} \oint_\gamma \underbrace{A \wedge A \wedge \cdots \wedge A}_r \right] \quad (2.23)$$

$$A = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} A_{ij} \omega_{ij} \quad (2.24)$$

$$\omega_{ij} = d \log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j} \quad (2.25)$$

$$A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)} \quad (2.26)$$

3 Geometric quantization and linking numbers

In the following sections, we shall consider zero-mode contributions to the abelian holonomy operator $\Theta_{R,\gamma}(z)$ by changing the topology of $\mathbf{CP}^1 = S^2$ to $T^2 = S^1 \times S^1$. This means that we impose a double periodicity condition on the complex variables z_i ($i = 1, 2, \dots, n$):

$$z_i \rightarrow z_i + m_i + n_i \tau \quad (3.1)$$

where m_i and n_i are integers and $\tau = \text{Re}\tau + i\text{Im}\tau$ is the modular parameter of the torus. We shall carry out this analysis by use of a geometric-quantization scheme developed in [10] for the $U(1)$ Chern-Simons theory on $S^1 \times S^1 \times \mathbf{R}$. In

this preparatory section, we briefly review this scheme and show its relation to linking numbers.

Holonomies of torus

Torus can be described in terms of two real coordinates ξ_1, ξ_2 with periodicity of $\xi_r \rightarrow \xi_r + m$ ($r = 1, 2$) where m is any integer. In other words, ξ_r take real values in $0 \leq \xi_r \leq 1$, with the boundary values 0, 1 being identical. Complex coordinates of torus can be parametrized as $z = \xi_1 + \tau \xi_2$ where $\tau \in \mathbf{C}$ is the modular parameter of the torus. This parametrization is equivalent to (3.1). Notice that we can absorb the real part of τ into ξ_1 without losing generality. In the following, we then assume $\text{Re}\tau = 0$, *i.e.*,

$$\tau = \text{Re}\tau + i\text{Im}\tau = i\text{Im}\tau. \quad (3.2)$$

There are two noncontractible cycles on torus, conventionally labeled as α and β cycles. Holonomies of torus can be defined along these cycles as

$$\int_{\alpha} \omega = 1, \quad \int_{\beta} \omega = \tau \quad (3.3)$$

where the holomorphic one-form $\omega = \omega(z)dz$ is a zero mode of the anti-holomorphic derivative $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$. The normalization of ω and its complex conjugate $\bar{\omega}$ is given by

$$\int dz d\bar{z} \bar{\omega} \wedge \omega = i2\text{Im}\tau. \quad (3.4)$$

In constructing a theory on torus, we need to take account of factors that involve ω and $\bar{\omega}$.

Let $a \in \mathbf{C}$ be a complex physical variable of the zero mode. As we discuss in a moment, an abelian gauge potential on torus can be parametrized solely by this complex variable a and its conjugate. Notice that these also satisfy the double periodicity

$$a \rightarrow a + m + n\tau, \quad \bar{a} \rightarrow \bar{a} + m + n\bar{\tau} \quad (3.5)$$

where m and n are integers which correspond to the winding numbers of α and β cycles, respectively.

Using the assumption (3.2), we can rewrite the holonomies (3.3) in a more convenient form:

$$\oint_{\alpha_r} \omega_s = (\text{Im}\tau) \epsilon_{rs}, \quad (3.6)$$

$$\omega_1 = (d\bar{z} - dz)/2i = -\text{Im}\tau d\xi_2, \quad (3.7)$$

$$\omega_2 = (\tau d\bar{z} - \bar{\tau} dz)/2i = \text{Im}\tau d\xi_1 \quad (3.8)$$

where ϵ_{rs} ($r, s = 1, 2$) denotes the rank-2 Levi-Civita symbol and α_1, α_2 correspond to the α and β cycles, respectively. Notice that we can set $\omega(z) = 1$ in (3.3) by identifying the α and β cycles with loop integrals along the variables ξ_1 and ξ_2 , respectively. Normalization for ω_1 and ω_2 is given by

$$\int dz d\bar{z} \frac{\omega_1}{\text{Im}\tau} \wedge \frac{\omega_2}{\text{Im}\tau} = 1. \quad (3.9)$$

We now reparametrize the complex physical variables as

$$a_1 = \bar{a} - a, \quad a_2 = \tau \bar{a} - \bar{\tau} a. \quad (3.10)$$

These are compatible with the choice of ω_1 and ω_2 . Under the transformations of (3.5), a_1 and a_2 vary as

$$\delta a_1 \rightarrow (-2i\text{Im}\tau)n, \quad \delta a_2 \rightarrow (2i\text{Im}\tau)m. \quad (3.11)$$

From (3.6) and (3.11), we find

$$\exp\left(\oint_{\alpha_2} \frac{\pi\omega_1}{\text{Im}\tau} \frac{\delta a_2}{\text{Im}\tau}\right) = e^{-i2\pi m}, \quad \exp\left(\oint_{\alpha_1} \frac{\pi\omega_2}{\text{Im}\tau} \frac{\delta a_1}{\text{Im}\tau}\right) = e^{-i2\pi n}. \quad (3.12)$$

Geometric quantization of the toric $U(1)$ Chern-Simons theory

We now consider geometric quantization of the $U(1)$ Chern-Simons theory, following the line of [8, 9] in a slightly different manner.

In the temporal gauge, Chern-Simons gauge potentials can be described by the spatial components. In terms of the real coordinates ξ_r ($r = 1, 2$) on torus, these can be parametrized as

$$\begin{aligned} A_{\xi_1} &= i\partial_{\xi_1}\theta + \frac{\pi\omega_2}{\text{Im}\tau} \frac{a_1}{\text{Im}\tau}, \\ A_{\xi_2} &= i\partial_{\xi_2}\theta + \frac{\pi\omega_1}{\text{Im}\tau} \frac{a_2}{\text{Im}\tau} \end{aligned} \quad (3.13)$$

where ∂_{ξ_r} denotes $\frac{\partial}{\partial \xi_r}$ and $\theta = \theta(\xi_1, \xi_2)$ is a function of ξ_1, ξ_2 . From (3.12), we can easily find that holonomies of A_{ξ_r} are invariant under the transformations of (3.5). Notice also that, under a certain gauge where θ is a constant, the

gauge potentials are parametrized purely by zero modes.¹ Thus, in terms of the complexified variables, the gauge potentials of the $U(1)$ Chern-Simons theory on $S^1 \times S^1 \times \mathbf{R}$ can be parametrized by

$$A_z = \frac{\pi\omega}{\text{Im}\tau} \bar{a}, \quad A_{\bar{z}} = \frac{\pi\bar{\omega}}{\text{Im}\tau} a. \quad (3.14)$$

In the program of geometric quantization, a “holomorphic” wavefunction $\Psi[A_{\bar{z}}]$ of the $U(1)$ Chern-Simons theory generally satisfies the so-called polarization condition

$$\left(\partial_a + \frac{1}{2} \partial_a K \right) \Psi[A_{\bar{z}}] = 0 \quad (3.15)$$

where K is a Kähler potential that is associated with the phase space of Chern-Simons theory in the $A_0 = 0$ gauge. The condition (3.15) leads to the specific form

$$\Psi[A_{\bar{z}}] = e^{-\frac{K}{2}} \psi[A_{\bar{z}}] \quad (3.16)$$

where $\psi[A_{\bar{z}}]$ is a purely (anti)holomorphic function of $A_{\bar{z}}$. In the present case, physical variables are given by a and \bar{a} . Thus we can express the holomorphic wavefunction (3.16) as

$$\Psi[A_{\bar{z}}] \equiv \Psi[a] = e^{-\frac{K(a, \bar{a})}{2}} f(a) \quad (3.17)$$

where $f(a)$ is a function of a . This is a function defined on torus. Thus it is natural to require the invariance of $f(a)$ under the transformation $a \rightarrow a + m + n\tau$. In the following, we shall show this relation by a suitable choice of the Kähler potential $K(a, \bar{a})$.

We first notice that there is an ambiguity in defining Kähler potentials. What is essential in physics in a geometric analysis is the Kähler form to start with. There are a number of Kähler potentials that leads to the same Kähler form. In the present case, form (3.4) and (3.14), we can define the Kähler form of zero modes as

$$\Omega^{(\tau)} = \frac{k}{2\pi} da \wedge d\bar{a} \int_{z, \bar{z}} \left(\frac{\pi\bar{\omega}}{\text{Im}\tau} \right) \wedge \left(\frac{\pi\omega}{\text{Im}\tau} \right) = i \frac{\pi k}{\text{Im}\tau} da \wedge d\bar{a} \quad (3.18)$$

where the integral is taken over $dzd\bar{z}$ and k is the level number associated to the abelian Chern-Simons theory. The corresponding Kähler potentials can

¹For non-abelian theories, it is not possible to express A_{ξ_1} , A_{ξ_2} entirely by zero modes; see [10] for details.

generally be expressed as

$$W(a, \bar{a}) = \frac{\pi k}{\text{Im}\tau} a\bar{a} + g(a) + h(\bar{a}) \quad (3.19)$$

where $g(a)$ and $h(\bar{a})$ are arbitrary functions of a and \bar{a} , respectively. Thus there are infinite choices for the Kähler potentials. We need to choose an appropriate one depending on specific purposes. This is what we are going to do now.

From our normalization (3.4) and (3.9), we find a relation

$$da \wedge d\bar{a} \int_{z, \bar{z}} \bar{\omega} \wedge \omega = da_1 \wedge da_2 \int_{z, \bar{z}} \frac{\omega_2}{\text{Im}\tau} \wedge \frac{\omega_1}{\text{Im}\tau} \quad (3.20)$$

where we use

$$\omega = \frac{\omega_2 - \tau\omega_1}{\text{Im}\tau}, \quad \bar{\omega} = \frac{\omega_2 - \bar{\tau}\omega_1}{\text{Im}\tau}. \quad (3.21)$$

In terms of a_1 and a_2 , the zero-mode Kähler form can then be expressed as

$$\begin{aligned} \Omega^{(\tau)} &= \frac{k}{2\pi} \left(\frac{\pi}{\text{Im}\tau} \right)^2 da \wedge d\bar{a} \int_{z, \bar{z}} \bar{\omega} \wedge \omega = \frac{k}{2\pi} \left(\frac{\pi}{\text{Im}\tau} \right)^2 (2i\text{Im}\tau) da \wedge d\bar{a} \\ &= \frac{k}{2\pi} \left(\frac{\pi}{\text{Im}\tau} \right)^2 da_1 \wedge da_2 \int_{z, \bar{z}} \frac{\omega_2}{\text{Im}\tau} \wedge \frac{\omega_1}{\text{Im}\tau} = -\frac{k}{2\pi} \left(\frac{\pi}{\text{Im}\tau} \right)^2 da_1 \wedge da_2. \end{aligned} \quad (3.22)$$

A Kähler potential corresponding to the second line in (3.22) can be written as

$$K(a, \bar{a}) = \frac{i\pi k}{2(\text{Im}\tau)^2} (\bar{a} - a)(\tau\bar{a} - \bar{\tau}a). \quad (3.23)$$

We shall choose this $K(a, \bar{a})$ as our Kähler potential for zero modes.

The symplectic potential corresponding to $\Omega^{(\tau)}$ can be expressed as

$$\begin{aligned} \mathcal{A}^{(\tau)} &= \frac{\pi k}{4(\text{Im}\tau)^2} \int_{z, \bar{z}} \left(\frac{\omega_2 a_1}{\text{Im}\tau} \wedge \frac{\omega_1}{\text{Im}\tau} da_2 - \frac{\omega_1 a_2}{\text{Im}\tau} \wedge \frac{\omega_2}{\text{Im}\tau} da_1 \right) \\ &= -\frac{\pi k}{4(\text{Im}\tau)^2} (a_1 da_2 + a_2 da_1). \end{aligned} \quad (3.24)$$

Notice that we can interpret $\mathcal{A}^{(\tau)}$ as a $U(1)$ gauge potential on torus. In this sense, the transformations (3.5) serve as gauge transformations. From (3.11) we find that a variation of $\mathcal{A}^{(\tau)}$ under $a \rightarrow a + m + n\tau$ is given by

$$\mathcal{A}^{(\tau)} \rightarrow \mathcal{A}^{(\tau)} + d\Lambda_{m,n} \quad (3.25)$$

$$\Lambda_{m,n} = -i \frac{\pi k}{2\text{Im}\tau} (ma_1 - na_2). \quad (3.26)$$

Gauge invariance of the wavefunction $\Psi[a]$ in (3.17) is then realized by imposing

$$e^{i\Lambda_{m,n}}\Psi[a] = \Psi[a + m + n\tau]. \quad (3.27)$$

This leads to the following relation

$$f(a) = e^{-i\pi kmn}f(a + m + n\tau). \quad (3.28)$$

Therefore the holomorphic function $f(a)$ is invariant under $a \rightarrow a + m + n\tau$, given that the level number k is quantized by even integers, *i.e.*,

$$k \in 2\mathbf{Z}. \quad (3.29)$$

This level-number quantization condition is well-known for the toric $U(1)$ Chern-Simons theory. Here we have reviewed this fact in the context of geometric quantization.

An inner product of the holomorphic wavefunctions can be expressed as

$$\langle 1|2 \rangle = \int d\mu(a, \bar{a}) e^{-K(a, \bar{a})} \overline{f_1(a)} f_2(a) \quad (3.30)$$

where $d\mu(a, \bar{a}) = [dad\bar{a}]$ denotes the integration measure of the zero-mode variables up to normalization and $\overline{f_1(a)}$ is the complex conjugate of the function $f_1(a)$. Action of the derivative $\frac{\partial}{\partial a}$ on $f(a)$ leads to a factor of $\frac{\pi k}{\text{Im}\tau}\bar{a}$. This is a natural consequence of geometric quantization with our Kähler form (3.18) that implies the “phase space” of the zero-mode variables is given by $(\frac{\pi k}{\text{Im}\tau}\bar{a}, a)$. Thus, regardless the choices of Kähler potentials, we always have the operative relation:

$$\frac{\pi k}{\text{Im}\tau}\bar{a} \leftrightarrow \frac{\partial}{\partial a}. \quad (3.31)$$

Notice also that the holomorphic wavefunction $\Psi[a]$ and its inner product (3.30) completely specify the toric $U(1)$ Chern-Simons theory as a topological quantum field theory.

Realization of linking numbers

The level-number quantization (3.29) is derived from the invariance of the holomorphic function $f(a)$ under $a \rightarrow a + m + n\tau$. In quantum theory, however, physical observables are given by the square of wavefunctions. Thus, in the present case, $|f(a)|^2$ is physically more interesting and, in this sense, we can relax the condition (3.29). This allows us to set the level number ($k \in \mathbf{N}$) to

$$k = 1. \quad (3.32)$$

If $k > 1$, we can absorb it into one of the winding numbers, say m . The “phase factor” $\exp(i\pi kmn)$ in (3.28) then becomes

$$(-1)^{mn} \equiv (-1)^{lk(\alpha^m, \beta^n)} \quad (3.33)$$

where we introduce the notation $lk(\alpha^m, \beta^n)$ because the value mn , as seen in a moment, can naturally be interpreted as a linking number of α and β cycles; remember that m and n respectively denote the winding numbers of these cycles on torus.

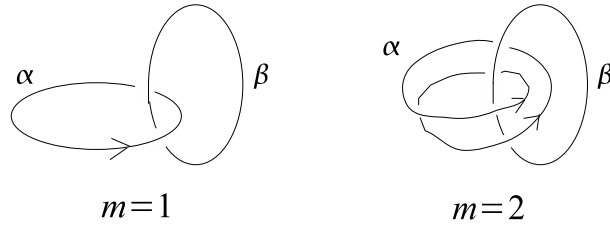


Figure 1: How the α and β cycles are entangled with each other in the cases of linking number $m = 1, 2$. The arrows correspond to the (positive) sings of m .

In defining the linking number, we can in fact fix one integer, say n , because we can always unravel one cycle so that the entanglement of two cycles can be measured by a winding number of one cycle around the other. We therefore fix the value of n to identity

$$n = 1. \quad (3.34)$$

Indeed, as seen in Figure 1, the linking number of α and β cycles can be defined by $m = lk(\alpha^m, \beta)$. The relation (3.28) is then reduces to

$$f(a) = (-1)^m f(a + m + i\text{Im}\tau). \quad (3.35)$$

4 Zero-mode contributions to the abelian holonomy operator

Construction of $\Theta_{R,\gamma}(z, a)$

We now apply the results in section 3 to the abelian holonomy operator $\Theta_{R,\gamma}(z)$ which has been defined in (2.23). *Our strategy is to incorporate multiple zero-mode variables a_i ($i = 1, 2, \dots, n$) into $\Theta_{R,\gamma}(z)$ by use of an abelian gauge theory.* Thus we first consider the covariantization of the comprehensive “photon” field A :

$$A \rightarrow \tilde{A} = A + A^{(\tau)} \quad (4.1)$$

where $A^{(\tau)}$ is an analog of A for zero modes. We can then define $A^{(\tau)}$ as

$$A^{(\tau)} = \sum_{i < j} A_{ij}^{(\tau)} \omega_{ij} \quad (4.2)$$

where ω_{ij} are the logarithmic one-forms given in (2.5) and $A_{ij}^{(\tau)}$ is a bialgebraic operator for zero modes. In the following, we shall determine possible forms of $A_{ij}^{(\tau)}$.

As discussed in (3.14), we can parametrize a holomorphic gauge field of zero modes as

$$A_z = \frac{\pi \bar{a}}{\text{Im} \tau} \omega. \quad (4.3)$$

From (3.31) we know that the factor $\frac{\pi \bar{a}}{\text{Im} \tau}$, acting on functions of a , can be replaced by a derivative operator ∂_a . Notice that the derivative operators do not have to obey the $SL(2, \mathbf{C})$ algebra. This is a crucial difference from the photon-creation operators $a_i^{(\pm)}$. In fact, by introducing a conjugate operator $\bar{\partial}_a$ which corresponds to \bar{a} , we find that these operators obey the following bosonic algebra:

$$[\partial_{a_i}, \bar{\partial}_{a_j}] = \delta_{ij}, \quad [\partial_{a_i}, \partial_{a_j}] = [\bar{\partial}_{a_i}, \bar{\partial}_{a_j}] = 0. \quad (4.4)$$

An analog of a number operator can be defined by

$$N_i = \bar{\partial}_{a_i} \partial_{a_i} \quad (4.5)$$

which satisfies

$$[N_i, \partial_{a_j}] = -\delta_{ij} \partial_{a_i}, \quad [N_i, \bar{\partial}_{a_j}] = \bar{\partial}_{a_i} \delta_{ij}. \quad (4.6)$$

In the context of geometric quantization, N_i corresponds to the area of the i -th phase space $(\frac{\pi \bar{a}_i}{\text{Im} \tau}, a_i)$. In this sense, we can also consider the bosonic algebra (4.4) as an analog of the Heisenberg algebra. As in the previous case, from this algebra it is natural for us to assume the form of $A_{ij}^{(\tau)}$ as

$$A_{ij}^{(\tau)} = N_i \otimes \partial_{a_j} + N_i \otimes \bar{\partial}_{a_j}. \quad (4.7)$$

In defining the holonomy operator for \tilde{A} , all what we need is to show the infinitesimal braid relations, *i.e.*,

$$[\tilde{A}_{ij}, \tilde{A}_{kl}] = 0 \quad (i, j, k, l \text{ are distinct}) \quad (4.8)$$

$$[\tilde{A}_{ij} + \tilde{A}_{jk}, \tilde{A}_{ik}] = 0 \quad (i, j, k \text{ are distinct}) \quad (4.9)$$

As in the case of A , the first relation is obvious since any commutators of two operators that have distinct numbering indices vanish. Thus it is sufficient to show the second relation and this can easily be checked as follows. We first expand $[\tilde{A}_{ij}, \tilde{A}_{ik}]$ as

$$\begin{aligned} [\tilde{A}_{ij}, \tilde{A}_{ik}] &= [A_{ij} + A_{ij}^{(\tau)}, A_{ik} + A_{ik}^{(\tau)}] \\ &= [A_{ij}, A_{ik}] + [A_{ij}, A_{ik}^{(\tau)}] + [A_{ij}^{(\tau)}, A_{ik}] + [A_{ij}^{(\tau)}, A_{ik}^{(\tau)}] \end{aligned} \quad (4.10)$$

Each of these terms can be evaluated by using the definition of a commutator for bialgebraic operators

$$\begin{aligned} [c_i \otimes d_i, c_j \otimes d_j] &= [c_i, c_j] \otimes d_i \otimes d_j \\ &\quad + c_i \otimes [d_i, c_j] \otimes d_j \\ &\quad + c_i \otimes c_j \otimes [d_i, d_j] \end{aligned} \quad (4.11)$$

where c_i and d_i ($i = 1, 2, \dots, n$) denote arbitrary operators; these are usual algebraic operators. As shown in [3], the first term vanishes:

$$\begin{aligned} [A_{ij}, A_{ik}] &= [a_i^{(+)} \otimes a_j^{(0)}, a_i^{(-)} \otimes a_k^{(0)}] + [a_i^{(-)} \otimes a_j^{(0)}, a_i^{(+)} \otimes a_k^{(0)}] \\ &= 2a_i^{(0)} \otimes a_j^{(0)} \otimes a_k^{(0)} - 2a_i^{(0)} \otimes a_j^{(0)} \otimes a_k^{(0)} = 0. \end{aligned} \quad (4.12)$$

Since N_i and $a_i^{(\pm)}$ commute, the second and third terms vanish. Further, from the definition (4.7), we can easily find

$$[A_{ij}^{(\tau)}, A_{ik}^{(\tau)}] = [N_i \otimes \partial_{a_j} + N_i \otimes \bar{\partial}_{a_j}, N_i \otimes \partial_{a_k} + N_i \otimes \bar{\partial}_{a_k}] = 0. \quad (4.13)$$

Similarly, we can easily find $[\tilde{A}_{jk}, \tilde{A}_{ik}] = 0$ by using $[a_k^{(0)}, \partial_{a_k}] = [a_k^{(0)}, \bar{\partial}_{a_k}] = 0$ and $[A_{jk}^{(\tau)}, A_{ik}^{(\tau)}] = 0$. The first equations are obvious since $a_k^{(0)}$ is independent of zero modes. The second relation can also be checked by

$$\begin{aligned} [A_{jk}^{(\tau)}, A_{ik}^{(\tau)}] &= [N_j \otimes \partial_{a_k} + N_j \otimes \bar{\partial}_{a_k}, N_i \otimes \partial_{a_k} + N_i \otimes \bar{\partial}_{a_k}] \\ &= N_j \otimes N_i \otimes [\partial_{a_k}, \bar{\partial}_{a_k}] + N_j \otimes N_i \otimes [\bar{\partial}_{a_k}, \partial_{a_k}] \\ &= 0. \end{aligned} \quad (4.14)$$

Therefore the infinitesimal braid relations (4.8), (4.9) indeed hold and we can properly define the holonomy operator for \tilde{A} :

$$\Theta_{R,\gamma}(z, a) = \text{Tr}_{R,\gamma} \text{P exp} \left[\sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \tilde{A} \wedge \cdots \wedge \tilde{A}}_r \right]. \quad (4.15)$$

Utilizing the form of (4.7), we can calculate $[\tilde{A}_{12}, \tilde{A}_{23}]$ as

$$\begin{aligned} [\tilde{A}_{12}, \tilde{A}_{23}] &= [A_{12}, A_{23}] + [A_{12}^{(\tau)}, A_{23}^{(\tau)}] \\ [A_{12}^{(\tau)}, A_{23}^{(\tau)}] &= N_1 \otimes \partial_{a_2} \otimes \partial_{a_3} + N_1 \otimes \partial_{a_2} \otimes \bar{\partial}_{a_3} \\ &\quad - N_1 \otimes \bar{\partial}_{a_2} \otimes \partial_{a_3} - N_1 \otimes \bar{\partial}_{a_2} \otimes \bar{\partial}_{a_3} \end{aligned} \quad (4.16)$$

where $[A_{12}, A_{23}]$ is given by (2.19). Thus, as in the case of (2.20), the exponents of (4.15) with path ordering P can be expressed as

$$\begin{aligned} &\text{P} \sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \tilde{A} \wedge \cdots \wedge \tilde{A}}_r \\ &= \sum_{r \geq 2} \oint_{\gamma} \tilde{A}_{12} \tilde{A}_{23} \cdots \tilde{A}_{r1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{r1} \\ &= \sum_{r \geq 2} \frac{1}{4^{r+1}} \sum_{(h_1, h_2, \dots, h_r)} (-1)^{h_1 + h_2 + \cdots + h_r} \\ &\quad \times \left(a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} + \partial_{a_1} \otimes \partial_{a_2} \otimes \cdots \otimes \partial_{a_r} \right. \\ &\quad \left. + (\text{terms involving } \bar{\partial}_i \text{'s}) \right) \\ &\quad \times \oint_{\gamma} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{r1} \end{aligned} \quad (4.17)$$

where, in analogy to (2.21), we define $N_1 \otimes \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} \otimes \partial_{a_1}$ as

$$\frac{1}{2} [\partial_{a_1}, N_1] \otimes \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} = \frac{1}{2} \partial_{a_1} \otimes \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r}. \quad (4.18)$$

The expression shows a natural extension of the original form (2.20). It gives a hybrid version of $a_i^{(\pm)}$'s and ∂_{a_i} 's. Thus we may consider the holonomy operator $\Theta_{R,\gamma}(z, a)$ with (4.7) as a natural generalization of $\Theta_{R,\gamma}(z)$ with zero mode variables. In practical calculations, however, we need to take account of the double periodicity of z_i 's and a_i 's. We shall analyze this point in a moment but before doing so let us consider another possibility for the choice of $A_{ij}^{(\tau)}$ which turns out to be more useful for the calculation of the zero-mode part of $\Theta_{R,\gamma}(z, a)$.

Notice that we make the terms that involve $\bar{\partial}_i$'s in (4.17) implicit since, as discussed in the previous section, any physical observables of zero modes are described by holomorphic functions in terms of a_i 's. In other words, the conjugate derivatives $\bar{\partial}_{a_i}$'s are auxiliary operators. Motivated by this fact, we can assume another form of $A_{ij}^{(\tau)}$:

$$A_{ij}^{(\tau)} = 1_i \otimes \partial_{a_j} . \quad (4.19)$$

where 1_i is an identity that acts on the i -th Fock space V_i . We may make the dimension of 1_i the same as that of the $SL(2, \mathbf{C})$ representation ρ ; thus this identity is equivalent to those that appear in (2.8). By construction, A_{ij} and $A_{ij}^{(\tau)}$ decouple to each other. In fact, $A_{ij}^{(\tau)}$ can be interpreted as a c -number operator since the constituents of $A_{ij}^{(\tau)}$ are now given by c -numbers. This is obvious from the fact that the derivative operators ∂_{a_i} can be replaced by $\frac{\pi \bar{a}_i}{\text{Im} \tau}$. In other words, there is no algebraic structure in $A_{ij}^{(\tau)}$ since it involves only the holomorphic derivatives. This means that the commutator $[A_{ij}^{(\tau)} + A_{jk}^{(\tau)}, A_{ik}^{(\tau)}]$ obviously vanishes. Thus the infinitesimal braid relations for $\tilde{A} = A + A^{(\tau)}$ automatically reduce to those of the “pure-gauge” potential A under the choice of (4.19). Therefore we can also define the holonomy operator in the form of (4.15), with $A_{ij}^{(\tau)}$ now redefined by (4.19).

As long as we follow our definition of path ordering P discussed in (2.16)-(2.22), the holonomy operator $\Theta_{R,\gamma}(z, a)$ automatically reduces to the pure-gauge operator $\Theta_{R,\gamma}(z)$. This can easily be seen from the fact that the commutator $[\tilde{A}_{12}, \tilde{A}_{23}]$, reduces to $[A_{12}, A_{23}]$ for the choice of (4.19), being in comparison with (4.16). Thus, in the present case, we need to relax the meaning of the path ordering P in order to extract zero-mode information out of $\Theta_{R,\gamma}(z, a)$. Remember that we have determined the path ordering P such that it is suitable for the bialgebraic operators of photons $A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}$ where $a_i^{(\pm)}$, $a_i^{(0)}$ obey the $SL(2, \mathbf{C})$ algebra. The zero-mode bialgebraic operators $A_{ij}^{(\tau)} = 1_i \otimes \partial_{a_j}$ are, however, essentially

given by the differential operators ∂_{a_j} which are free from the $SL(2, \mathbb{C})$ algebra. Thus it is natural to relax the meaning of the path ordering P for the calculation of zero-mode information out of the holonomy operator if we stick to the definition (4.19). (The change of algebraic properties in $\Theta_{R,\gamma}(z, a)$ also suggests that the zero-mode part of $\Theta_{R,\gamma}(z, a)$ is no longer a holonomy of conformal invariance but rather that of scale invariance.) In the following, we shall redefine the path ordering P such that we can suitably extract the zero-mode part of $\Theta_{R,\gamma}(z, a)$, which we shall denote by $\Theta_{R,\gamma}^{(\tau)}(z, a)$ from here on, with $A_{ij}^{(\tau)}$ having the form of (4.19).

Extraction of $\Theta_{R,\gamma}^{(\tau)}(z, a)$

Partly related to the redefinition of the path ordering, there is a crucial condition for the calculation of $\Theta_{R,\gamma}(z, a)$, *i.e.*, the double periodicity condition (3.1) for z_i 's. Remember that the complex coordinate on torus is parametrized by $z = \xi_1 + \tau\xi_2$ where $0 \leq \xi_r \leq 1$ ($r = 1, 2$), with $\xi_r = 0$ and $\xi_r = 1$ being identified. Since we set $\text{Re}\tau$ to zero, the complex coordinate is then expressed as $z = \xi_1 + i\text{Im}\tau\xi_2$. This means that z can be parametrized by ξ_1 if we suitably scale $\text{Im}\tau$ or absorb ξ_2 into the definition of $\text{Im}\tau$. Namely the imaginary part of z can be controlled by $\text{Im}\tau$. Thus, in our settings, z is essentially parametrized by $0 \leq \xi_1 \leq 1$.

For an n -particle system, we have n such parameters z_i ($i = 1, 2, \dots, n$). Owing to the braid trace in $\Theta_{R,\gamma}(z, a)$, the holonomy operator preserves permutation invariance over the numbering index i . Thus, without losing generality, we can impose the ordering condition

$$0 \leq z_1 \leq z_2 \leq \dots \leq z_n \leq 1 \quad (4.20)$$

where we consider z_i 's as real variables. In the following, we assume this condition unless mentioned otherwise. The number of elements for the permutation or the braid group \mathcal{B}_n is $n - 1$. Thus we can fix one of z_i 's, say $z_1 = 0$. Since the boundary values are identical, this means that we can further fix z_n to $z_n = 1$. Namely, we shall impose

$$z_1 = 0, \quad z_n = 1 \quad (4.21)$$

on top of the ordering condition (4.20). This suggests that the path integral in $\Theta_{R,\gamma}(z, a)$ can be carried out along the path γ defined by the line segment

$$\gamma = [0, 1] \quad (4.22)$$

embedded in a complex plane. This does not contradict with the fact that the path γ is defined on the physical configuration space $\mathcal{C} = \mathbf{C}^n/\mathcal{S}_n$ since z_i 's are now all on $\gamma = [0, 1]$. Because the boundaries of the line segment are identical, the path γ also represents a closed path.

The condition (4.21) is very stringent. For example, applying this to the logarithmic one-form $\omega_{ij} = d \log(z_i - z_j)$, we find

$$\omega_{1j} = \frac{dz_1 - dz_j}{z_1 - z_j} = \frac{dz_j}{z_j} \equiv \omega_j^{(0)}, \quad (4.23)$$

$$\omega_{nj} = \frac{dz_n - dz_j}{z_n - z_j} = \frac{-dz_j}{1 - z_j} \equiv \omega_j^{(1)}. \quad (4.24)$$

Our strategy is to express the zero-mode part of the holonomy operator, denoted by $\Theta_{R,\gamma}^{(\tau)}(z, a)$, in terms of $\omega_j^{(0)}$ and $\omega_j^{(1)}$. As in (2.17), the exponent of $\Theta_{R,\gamma}(z, a)$ can be expanded as

$$\sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \tilde{A} \wedge \cdots \wedge \tilde{A}}_r = \sum_{r \geq 2} \oint_{\gamma} \sum_{(i < j)} \tilde{A}_{i_1 j_1} \tilde{A}_{i_2 j_2} \cdots \tilde{A}_{i_r j_r} \bigwedge_{k=1}^r \omega_{i_k j_k} \quad (4.25)$$

where γ is specified by (4.22) and $(i < j)$ means that the set of indices $(i_1, j_1, \dots, i_r, j_r)$ are ordered such that $1 \leq i_1 < j_1 \leq r, \dots, 1 \leq i_r < j_r \leq r$. As mentioned earlier, action of the path ordering P which has been defined in (2.18) does not give zero-mode information. In order to obtain the zero-mode information, we now introduce a new path ordering $P^{(\tau)}$ defined by

$$\begin{aligned} & P^{(\tau)} \sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \tilde{A} \wedge \cdots \wedge \tilde{A}}_r \\ &= \sum_{r \geq 2} \oint_{\gamma} \tilde{A}_{12} \tilde{A}_{13} \cdots \tilde{A}_{1r} \tilde{A}_{r1} \omega_{12} \wedge \omega_{13} \wedge \cdots \wedge \omega_{1r} \wedge \omega_{r1} \\ &= \sum_{r \geq 2} \oint_{\gamma} \tilde{A}_{12} \tilde{A}_{13} \cdots \tilde{A}_{1r} \tilde{A}_{r1} \omega_2^{(0)} \wedge \omega_3^{(0)} \wedge \cdots \wedge \omega_r^{(0)} \wedge \omega_1^{(1)}. \end{aligned} \quad (4.26)$$

The symbol $P^{(\tau)}$ means that, on top of the initial condition $(i < j)$, the indices $(i_1, j_1, \dots, i_r, j_r)$ are further constrained by $i_1 = i_2 = \cdots = i_{r-1} = 1$, $i_r = r$, and $2 \leq j_1 < j_2 < \cdots < j_r \leq r+1$ where $r+1$ is to be identified with 1. The ordering rule for j 's is the same as that of P ; the difference lies in the specific fixing of i 's. Notice that these extra conditions automatically lead to the above expression (4.26).

We now argue that the new path ordering $P^{(\tau)}$ in (4.26) automatically leads to the zero-mode holonomy operator. As mentioned below (4.19),

$A_{ij}^{(\tau)} = 1_i \otimes \partial_{a_j}$ can be treated as a c -number operator which decouples with $A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}$. Thus the factor of $\tilde{A}_{12}\tilde{A}_{13}\cdots\tilde{A}_{1r}\tilde{A}_{r1}$ in (4.26) is split into the photon part $A_{12}A_{13}\cdots A_{1r}A_{r1}$ and the zero-mode part $A_{12}^{(\tau)}A_{13}^{(\tau)}\cdots A_{1r}^{(\tau)}A_{r1}^{(\tau)}$. The vanishing of the photon-part can be shown by the commutation relation

$$\begin{aligned} [A_{12}, A_{13}] &= [a_1^{(+)} \otimes a_2^{(0)} + a_1^{(-)} \otimes a_2^{(0)}, a_1^{(+)} \otimes a_3^{(0)} + a_1^{(-)} \otimes a_3^{(0)}] \\ &= 2a_1^{(0)} \otimes a_2^{(0)} \otimes a_3^{(0)} - 2a_1^{(0)} \otimes a_2^{(0)} \otimes a_3^{(0)} = 0. \end{aligned} \quad (4.27)$$

On the other hand, the zero-mode part $A_{12}^{(\tau)}A_{13}^{(\tau)}\cdots A_{1r}^{(\tau)}A_{r1}^{(\tau)}$ gives a c -number operator acting on the Hilbert space $V^{\otimes r} = V_1 \otimes V_2 \otimes \cdots \otimes V_r$ for zero modes. Thus we can interpret this part as an coefficient of the loop integral in (4.26). Explicitly this factor can be calculated as

$$A_{12}^{(\tau)}A_{13}^{(\tau)}\cdots A_{1r}^{(\tau)}A_{r1}^{(\tau)} = \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} \otimes \partial_{a_1}. \quad (4.28)$$

This statement is intuitively correct but is not mathematically rigorous since, if we treat $A_{ij}^{(\tau)}$ as a bialgebraic operator which we do in defining the (de)coupling between A_{ij} and $A_{ij}^{(\tau)}$, the commutator of $A_{ij}^{(\tau)}$'s obviously vanishes and, as in (4.27), we can argue that the zero-mode part of (4.26) also becomes zero. This problem originates from the fact that the physical operators in holonomy formalism obey the $SL(2, \mathbf{C})$ algebra. For the zero-mode variables, however, there are no nontrivial algebraic symmetries built-in as long as we use only the holomorphic part of the variables. In this sense, the zero-mode holonomy operator does not represent a holonomy operator of conformal invariance but rather scale invariance. Thus it may be necessary to consider the commutator of $A_{ij}^{(\tau)}$'s in constructing the full holonomy of $\tilde{A} = A + A^{(\tau)}$. But if one is interested in purely the holonomy of $A^{(\tau)}$, it is sufficient to interpret $A_{ij}^{(\tau)}$ as a c -number operator.

Having said these rather intuitive justifications, we now present a more satisfactory solution to the above problem, that is, the problem can be fixed by imposing an antisymmetric property on the derivative operators. Since derivatives are commutative, it seems impossible to impose such a condition. However, in the calculatory process in (4.26) it can be done by introducing a coupling of the derivative operator ∂_{a_j} with a Grassmann variable η_j . This variable is a ghost variable that appears only in the middle of calculation and will be integrated out at the end of calculation. The operator $A_{ij}^{(\tau)}$ is then rewritten as

$$A_{ij}^{(\tau)} = 1_i \otimes \int d\eta_j \eta_j \partial_{a_j} = 1_i \otimes \partial_{a_j} \quad (4.29)$$

where η_j is the Grassmann variable. The final expression of $A_{ij}^{(\tau)}$ remains the same but the above redefinition means that we make it a rule to carry out the Grassmann integral at the end of calculation. Thus with an introduction of the ghosts, we can calculate the quantity $A_{12}^{(\tau)} A_{13}^{(\tau)}$ as

$$\begin{aligned} A_{12}^{(\tau)} A_{13}^{(\tau)} &= \int d\eta_3 d\eta_2 \, 1_1 \otimes \partial_{a_2} \otimes \partial_{a_3} \, \eta_2 \eta_3 \\ &= \frac{1}{2} \int d\eta_3 d\eta_2 \, 1_1 \otimes \partial_{a_2} \otimes \partial_{a_3} \, (\eta_2 \eta_3 - \eta_3 \eta_2) \\ &= 1_1 \otimes \partial_{a_2} \otimes \partial_{a_3} . \end{aligned} \quad (4.30)$$

Similarly the factor the (4.28) can be expressed as

$$\begin{aligned} A_{12}^{(\tau)} A_{13}^{(\tau)} \cdots A_{1r}^{(\tau)} A_{r1}^{(\tau)} &= \int [d\eta] \, \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} \otimes \partial_{a_1} \, \eta_2 \eta_3 \cdots \eta_r \eta_1 \\ &= \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} \otimes \partial_{a_1} \end{aligned} \quad (4.31)$$

where $[d\eta] = d\eta_1 d\eta_r d\eta_{r-1} \cdots d\eta_2$. This shows that the use of (4.29) guarantees that the zero-mode part of (4.26) gives non-vanishing contributions even though ∂_{a_i} is a c -number operator. Thus the expression (4.26) can further be calculated as

$$\begin{aligned} &P^{(\tau)} \sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \cdots \wedge \tilde{A}}_r \\ &= \sum_{r \geq 2} \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} \otimes \partial_{a_1} \oint_{\gamma} \omega_2^{(0)} \wedge \omega_3^{(0)} \wedge \cdots \wedge \omega_r^{(0)} \wedge \omega_1^{(1)} . \end{aligned} \quad (4.32)$$

As discussed above, this result can also be obtained by direct use of (4.28). The introduction of the ghost variables is redundant in this sense but they are necessary if one is interested in how the zero-mode contributions incorporate into the abelian holonomy operators of conformal field theory. In what follows we are interested in only the zero-mode part of the holonomy operator. Thus we shall leave these calculatory issues aside in the rest of the present paper.

Integral representation of Riemann's zeta function

We now consider the integral part of the expression (4.32). This integral can be understood as an iterated integral of the following one-forms:

$$\omega^{(1)} = \frac{dt}{1-t} \equiv f_1(t) dt \quad (4.33)$$

$$\omega^{(0)} = \frac{dt}{t} \equiv f_0(t)dt \quad (4.34)$$

where $t \in \gamma = [0, 1]$. It is known that the polylogarithm function $\text{Li}_k(z)$ can be represented by an iterated integral in general [1]. For the case of $\text{Li}_2(1)$ this can be given by

$$\int_{\gamma} \omega^{(1)} \omega^{(0)} = \int_0^1 \frac{\log(1-t)}{t} dt = \text{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (4.35)$$

where we use the definition of the iterated integral

$$\int_{\gamma} \omega^{(1)} \omega^{(0)} = \int_0^1 \left(\int_0^{t_0} f_1(t_1) dt_1 \right) f_0(t_0) dt_0. \quad (4.36)$$

In general, higher-order iterated integrals in terms of one-forms

$$\omega_i = f_i(t)dt \quad (i = 1, 2, \dots, n), \quad t \in \gamma, \quad (4.37)$$

are defined by

$$\int_{\gamma} \omega_1 \omega_2 \cdots \omega_n = \int_{\Delta_n} f_1(t_1) f_2(t_2) \cdots f_n(t_n) dt_1 dt_2 \cdots dt_n \quad (4.38)$$

where Δ_n is given by

$$\Delta_n = \{(t_1, t_2, \dots, t_n) \in \mathbf{R}^n \mid 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1\}. \quad (4.39)$$

Using the definition, we can generalize the expression (4.35) to obtain

$$\text{Li}_k(1) = \int_0^1 \omega^{(1)} \underbrace{\omega^{(0)} \omega^{(0)} \cdots \omega^{(0)}}_{k-1} = \sum_{n \geq 1} \frac{1}{n^k} = \zeta(k). \quad (4.40)$$

where $\zeta(k)$ is Riemann's zeta function.

Comparing our construction of $\oint_{\gamma} \omega_2^{(0)} \wedge \omega_3^{(0)} \wedge \cdots \wedge \omega_r^{(0)} \wedge \omega_1^{(1)}$ along with (4.20)-(4.24) to the above definition of iterated integrals, we find that the loop integral can be calculated as

$$\begin{aligned} & \oint_{\gamma} \omega_2^{(0)} \wedge \omega_3^{(0)} \wedge \cdots \wedge \omega_r^{(0)} \wedge \omega_1^{(1)} \\ &= (-1)^{r-1} \oint_{\gamma} \omega_1^{(1)} \wedge \omega_2^{(0)} \wedge \omega_3^{(0)} \wedge \cdots \wedge \omega_r^{(0)} \\ &= (-1)^r \int_{\gamma} \omega^{(1)} \underbrace{\omega^{(0)} \omega^{(0)} \cdots \omega^{(0)}}_{r-1} = (-1)^r \zeta(r) \end{aligned} \quad (4.41)$$

where we follow the notation of (4.38) in the last line. The zero-mode holonomy operator is then expressed as

$$\Theta_{R,\gamma}^{(\tau)}(z, a) = \text{Tr}_{R,\gamma} P^{(\tau)} \exp \left[\sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \tilde{A} \wedge \cdots \wedge \tilde{A}}_r \right] \quad (4.42)$$

where an explicit form of the exponent is given by

$$P^{(\tau)} \sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \cdots \wedge \tilde{A}}_r = \sum_{r \geq 2} \partial_{a_1} \otimes \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} (-1)^r \zeta(r) . \quad (4.43)$$

In the abelian case, the trace $\text{Tr}_{R,\gamma}$ is represented by a braid trace. As in the expression (2.22), this is given by a sum over the permutation of the numbering indices. We shall recapitulate the final form $\Theta_{R,\gamma}^{(\tau)}(z, a) \equiv \Theta_{R,\gamma}^{(\tau)}(a)$ at the end of this section (see (4.51)).

Generalization of linking numbers

The vacuum expectation value (vev) of the zero-mode holonomy operator $\Theta_{R,\gamma}^{(\tau)}(z, a)$ or that of $\Theta_{R,\gamma}(z, a)$ in general can be considered as a holomorphic function of (a_1, a_2, \cdots, a_n) . Thus, if we consider gauge transformations of these variables, *i.e.*,

$$a_i \rightarrow a_i + m_i + i \text{Im} \tau , \quad (4.44)$$

we can certainly obtain analogs of linking numbers m_i ($i = 1, 2, \cdots, n$) out of $\Theta_{R,\gamma}(z, a)$; in what follows we shall use $\Theta_{R,\gamma}(z, a)$ for simplicity but this can always be replaced by $\Theta_{R,\gamma}^{(\tau)}(z, a)$. The corresponding holomorphic wavefunction, which is analogous to (3.17), is then written as

$$\Xi[\tilde{A}] = \exp \left(- \sum_{i=1}^n \frac{K(a_i, \bar{a}_i)}{2} \right) \langle \Theta_{R,\gamma}(z; a_1, a_2, \cdots, a_n) \rangle \quad (4.45)$$

where we make the a_i -dependence explicit. The bracket $\langle \cdot \rangle$ indicates that operators inbetween are evaluated at the vacuum state of the system which is define on the Hilbert space $V^{\otimes n}$ for zero modes. The Kähler potential $K(a_i, \bar{a}_i)$ is defined in (3.23). The polarization conditions for $\Xi[\tilde{A}]$ are given by

$$\left(\partial_{a_i} + \frac{1}{2} \partial_{a_i} K(a_i, \bar{a}_i) \right) \Xi[\tilde{A}] = 0 \quad (4.46)$$

for arbitrary i 's.

The holonomy operator gives a monodromy representation of the KZ equation. Thus it provides a general solution to the gauged KZ equation

$$(d - \tilde{A})\Psi(z; a_1, a_2, \dots, a_n) = 0 \quad (4.47)$$

which is a differential equation analogous to (2.10) with $\tilde{A} = A + A^{(\tau)}$. This KZ equation is holomorphic in a_i 's, while the polarization condition (4.46) is not. Thus, although these equations have similar structures, there is no direct way to connect them. Note that z_i and a_i are both coordinates on a torus, however, the former is a coordinate of the Riemann surface on which two-dimensional conformal field theory is defined, while the latter denotes a physical variable of zero modes on torus. Thus physical meaning of the complex variables z_i , a_i are distinct. This also explains the qualitative difference between the equations (4.46) and (4.47).

Now, applying the result of (3.35), we find the relation

$$\langle \Theta_{R,\gamma}(z, a) \rangle = (-1)^{m_1+m_2+\dots+m_n} \langle \Theta_{R,\gamma}(z, a + m + i\text{Im}\tau) \rangle \quad (4.48)$$

where m_i are integers and $\Theta_{R,\gamma}(z, a + m + i\text{Im}\tau)$ is defined by

$$\Theta_{R,\gamma}(z; a_1 + m_1 + i\text{Im}\tau, a_2 + m_2 + i\text{Im}\tau, \dots, a_n + m_n + i\text{Im}\tau). \quad (4.49)$$

As in the previous case, this relation can be checked by the gauge invariance of the holomorphic wavefunction $\Xi[\tilde{A}] = \Xi(z, a)$ under $a_i \rightarrow a_i + m_i + n_i\tau$ with $n_i = 1$, $\text{Re}\tau = 0$:

$$e^{i\Lambda_{m_i,1}}\Xi(z, a) = \Xi(z, a + m + i\text{Im}\tau) \quad (4.50)$$

where $\Lambda_{m_i,1}$ is defined in (3.26). Notice that, as it will be obvious from the context, a_1 and a_2 in (3.26) do not denote zero-mode variables for $i = 1, 2$ but correspond to the complex variables defined in (3.10) for arbitrary i 's.

The relation (4.48) gives a straightforward generalization of (3.35). This means that we can consider the sum $m_1 + m_2 + \dots + m_n$ in (4.48) as a generalized linking number. As discussed earlier, we can interpret m_i as a linking number of the α and β cycles of the i -th torus on which the zero-mode variable a_i is defined. The fact that we fix n_i to 1 for arbitrary i 's suggests that we may use a common β cycle for each of m_i 's (see Figure 2).

Summary

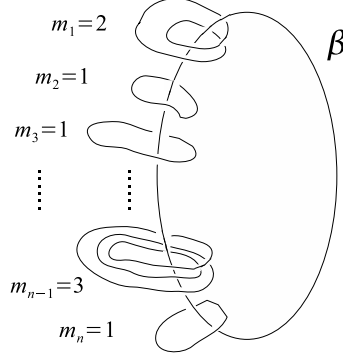


Figure 2: How α cycles entangle with the common β cycle. The sum $m_1 + m_2 + \cdots + m_n$ in (4.48) can be interpreted as a generalization of linking numbers in the abelian holonomy formalism.

As a summary of this section, we recapitulate the resultant expression for the zero-mode holonomy operator $\Theta_{R,\gamma}^{(\tau)}(z, a)$ below.

$$\begin{aligned}
\Theta_{R,\gamma}^{(\tau)}(z, a) &= \text{Tr}_{R,\gamma} P^{(\tau)} \exp \left[\sum_{r \geq 2} \oint_{\gamma} \underbrace{\tilde{A} \wedge \tilde{A} \wedge \cdots \wedge \tilde{A}}_r \right] \\
&= \text{Tr}_{R,\gamma} \exp \left[\sum_{r \geq 2} \partial_{a_1} \otimes \partial_{a_2} \otimes \partial_{a_3} \otimes \cdots \otimes \partial_{a_r} (-1)^r \zeta(r) \right] \\
&= \exp \left[\sum_{r \geq 2} \sum_{\sigma \in \mathcal{S}_{r-1}} \partial_{a_1} \otimes \partial_{a_{\sigma_2}} \otimes \partial_{a_{\sigma_3}} \otimes \cdots \otimes \partial_{a_{\sigma_r}} (-1)^r \zeta(r) \right] \\
&\equiv \Theta_{R,\gamma}^{(\tau)}(a)
\end{aligned} \tag{4.51}$$

where σ denote permutations of the numbering elements $\{2, 3, \cdots, r\}$, *i.e.*, $\sigma = \begin{pmatrix} 2 & 3 & \cdots & r \\ \sigma_2 & \sigma_3 & \cdots & \sigma_r \end{pmatrix}$. ∂_{a_i} ($i = 1, 2, \cdots, r$) are the derivative operators with respect to the zero-mode variables a_i . These operators appear in the definition of the comprehensive gauge field \tilde{A} :

$$\tilde{A} = A + A^{(\tau)}, \tag{4.52}$$

$$A^{(\tau)} = \sum_{1 \leq i < j \leq n} A_{ij}^{(\tau)} \omega_{ij}, \quad (4.53)$$

$$A_{ij}^{(\tau)} = 1_i \otimes \partial_{a_j} \quad (4.54)$$

where A is defined in (2.24)-(2.26). Gauge transformations of a_i , *i.e.*, $a_i \rightarrow a_i + m_i + i\text{Im}\tau$, in the vacuum expectation value of $\Theta_{R,\gamma}^{(\tau)}(a)$ leads to the relation

$$\langle \Theta_{R,\gamma}^{(\tau)}(a + m + i\text{Im}\tau) \rangle = \prod_{i=1}^n (-1)^{m_i} \langle \Theta_{R,\gamma}^{(\tau)}(a) \rangle \quad (4.55)$$

where the argument in the left-hand side means the same as that in (4.49) and $m_i \in \mathbf{Z}$ ($i = 1, 2, \dots, n$) can be interpreted as a linking number between the α_i cycle and the common β cycle (see Figure 2). As mentioned in the introduction, the concept of linking numbers can be related to the Legendre symbol of elementary number theory. Motivated by this fact, in the next section we shall consider $\Theta_{R,\gamma}^{(\tau)}(a)$ in a space of finite field \mathbf{F}_p where p is an odd prime number.

5 Application to the elementary theory of numbers

In this section we consider application of the zero-mode holonomy operator to a space of finite field \mathbf{F}_p . *As mentioned in the introduction, our strategy is to use Morishita's result (1.1) on the analogy between knots and primes.* We first review this result and related materials in number theory which are of direct relevance to later discussions.

Linking numbers, Legendre symbols and Jacobi symbols

For convenience and refreshment, we start writing down the result (1.1) again [15]:

$$(-1)^{lk(q,p)} = \left(\frac{q^*}{p} \right) \quad (5.1)$$

where $lk(q,p)$ is an analog of a linking number (mod 2) of two distinct odd primes, p, q , and $\left(\frac{q^*}{p} \right)$ is the Legendre symbol defined in (1.2). The value of q^* is given by $q^* = (-1)^{\frac{q-1}{2}} q$, *i.e.*,

$$q^* = \left(\frac{-1}{q} \right) q = \begin{cases} q & q \equiv 1 \pmod{4}; \\ -q & q \equiv 3 \pmod{4}. \end{cases} \quad (5.2)$$

The Legendre symbol satisfies the reciprocity law

$$\left(\frac{q^*}{p}\right) = \left(\frac{p}{q}\right). \quad (5.3)$$

In terms of $lk(p, q)$, this can also be written as

$$lk(q, p) = lk(p, q^*) = lk(p^*, q). \quad (5.4)$$

Now, denoting the Legendre symbol $\left(\frac{q^*}{p}\right)$ by $\lambda_p(q^*)$, we can express the Gauss sum as

$$\hat{\lambda}_p = \sum_{x=1}^{p-1} \lambda_p(x) e^{i \frac{2\pi}{p} x} \quad (5.5)$$

where $x \in \mathbf{F}_p^\times = \mathbf{F}_p - \{0\}$. It is well-known that the Gauss sum becomes

$$\hat{\lambda}_p = \sqrt{p^*} = \begin{cases} \sqrt{p} & p \equiv 1 \pmod{4}; \\ i\sqrt{p} & p \equiv 3 \pmod{4}. \end{cases} \quad (5.6)$$

For example, $\hat{\lambda}_3$ and $\hat{\lambda}_5$ can be calculated as

$$\begin{aligned} \hat{\lambda}_3 &= e^{i \frac{2\pi}{3}} - e^{i \frac{4\pi}{3}} = i\sqrt{3}, \\ \hat{\lambda}_5 &= e^{i \frac{2\pi}{5}} - e^{i \frac{4\pi}{5}} - e^{i \frac{6\pi}{5}} + e^{i \frac{8\pi}{5}} = \sqrt{5}. \end{aligned} \quad (5.7)$$

This means that the Gauss sum takes a value of complex number and that we can interpret the Legendre symbol as a map $\lambda_p : \mathbf{F}_p^\times \rightarrow \mathbf{C}$. The Gauss sum (5.5) can then be interpreted as a Fourier transform of $\lambda_p(x)$ in a space of mod p [16]. This interpretation is interesting but there is a caveat. Namely, we may need to deal with a sum over all odd primes in order to calculate the inverse Fourier transform, which is practically impossible. However, calculation of the Gauss sum (5.5), *per se*, does not involve such a sum and it will be useful to consider (5.5) as a Fourier transform particularly in applying a field-theoretic approach to elementary number theory. In this context, the “phase space” of interest is given by $\left(\frac{2\pi}{p}, x\right)$ with $x \in \mathbf{F}_p^\times$. In a language of quantum field theory, this suggests that the Legendre symbol $\lambda_p(x)$ can be interpreted as an operator in an x -space representation, *i.e.*, an operator in a space of mod p . In the same sense, the Gauss sum $\hat{\lambda}_p$ can be regarded as a conjugate operator, that is, we can interpret $\hat{\lambda}_p$ as an operator in a space of $\frac{2\pi}{p}$ which is conjugate to the space of \mathbf{F}_p^\times . Thus it is presumably natural to interpret $\hat{\lambda}_p$ as an operator that is relevant to creation of primes.

We shall follow this idea later in application of the zero-mode holonomy operator to number theory.

A generalization of the Legendre symbol does exist and it is called the Jacobi symbol. Let N be an odd positive integer whose prime factorization is given by

$$N = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} . \quad (5.8)$$

Then the Jacobi symbol is defined by

$$\left(\frac{y}{N} \right) = \left(\frac{y}{p_1} \right)^{e_1} \left(\frac{y}{p_2} \right)^{e_2} \cdots \left(\frac{y}{p_s} \right)^{e_s} \quad (5.9)$$

where y denotes an arbitrary integer. In terms of $\lambda_p(y)$, this can also be expressed as

$$\left(\frac{y}{N} \right) = \lambda_{p_1}(y)^{e_1} \lambda_{p_2}(y)^{e_2} \cdots \lambda_{p_s}(y)^{e_s} := \eta_N(y) . \quad (5.10)$$

It is known that the Jacobi symbol also satisfies the supplementary and reciprocity laws which are essentially the same as those of the Legendre symbol (see, *e.g.*, [16]). In number theory, the Jacobi symbols are particularly useful in calculation of the Legendre symbol $\left(\frac{x}{p} \right)$ where p is a large prime number.

Correspondence between knots and primes

We now consider the zero-mode holonomy operator $\Theta_{R,\gamma}^{(\tau)}(a)$ in a space of \mathbf{F}_p^\times , utilizing the above results. As discussed above, quantum theoretically the Legendre symbol can be interpreted as an operator in this space. On the other hand, the operative construction of $\Theta_{R,\gamma}^{(\tau)}(a)$ is summarized in (4.51) and its vacuum expectation values satisfy the relation (4.55). Thus a natural way to carry out our analysis is to interpret the factor of $(-1)^{m_i}$ in (4.55) as a Legendre symbol.

As discussed in (3.33), a linking number of the α_i and β_i cycles can be defined by

$$m_i n_i = lk(\alpha_i^{m_i}, \beta_i^{n_i}) \quad (5.11)$$

where m_i and n_i are the winding numbers of α_i and β_i cycles, respectively. We make the latter being fixed to $n_i = 1$. Furthermore, as shown in Figure 2, we have made β_i identical for arbitrary i 's. Thus in our settings the winding number m_i can be written as

$$m_i = lk(\alpha_i^{m_i}, \beta) . \quad (5.12)$$

In this section, we further impose

$$m_i = 1 \tag{5.13}$$

so that we can single out the linking number of the α_i cycle and the common β cycle in the following form:

$$(-1)^{m_i}|_{m_i=1} = (-1)^{lk(\alpha_i, \beta)}. \tag{5.14}$$

According to Morishita's analogies between knots and primes, the α_i and β cycles correspond to odd prime numbers. Thus we may express the above quantity as

$$\begin{aligned} (-1)^{lk(\alpha_i, \beta)} &= \lambda_\beta(\alpha_i^*) = \lambda_{\alpha_i}(\beta) \\ &= (-1)^{lk(\beta, \alpha_i^*)} = \lambda_{\alpha_i^*}(\beta^*) = \lambda_\beta(\alpha_i^*) \\ &= (-1)^{lk(\beta^*, \alpha_i)} = \lambda_{\alpha_i}(\beta) = \lambda_{\beta^*}(\alpha_i) \end{aligned} \tag{5.15}$$

where we use (5.4). These relations hold when α_i and β are all odd prime numbers. Notice, however, that we are going to consider the Legendre symbol as a *function* of $x \in \mathbf{F}_p^\times$. As we mentioned earlier, we shall consider the Legendre symbol as an *operator* of $x \in \mathbf{F}_p^\times$ eventually (see (5.27)); here we shall make a classical analysis for the moment. Since the elements of \mathbf{F}_p^\times include non-prime numbers, the relations (5.15) are not quite applicable to our settings but they do suggest that we have several (essentially two, as discussed below) interpretations to the Legendre symbols in terms of the correspondence between knots and primes. For example, using the relation $(-1)^{lk(\alpha_i, \beta)} = \lambda_{\alpha_i}(\beta)$ in (5.15), one can interpret α_i as an odd prime number and β as an integer in $\mathbf{F}_{\alpha_i}^\times$. Another example is to use the relation $(-1)^{lk(\beta^*, \alpha_i)} = \lambda_{\beta^*}(\alpha_i)$ and to interpret α_i as an integer in $\mathbf{F}_{\beta^*}^\times$ and β as an odd prime number.² By a suitable choice of α_i and β , we can in fact classify the Legendre symbols of (5.15) into two types, *e.g.*, $\lambda_{\alpha_i}(\beta)$ and $\lambda_{\beta^*}(\alpha_i)$, depending on the ordering of α_i (or α_i^*) and β (or β^*).

Using the interpretation of $(-1)^{lk(\alpha_i, \beta)} = \lambda_{\alpha_i}(\beta)$, we now illustrate a simple connection between the Jacobi symbol and the vacuum expectation values of the zero-mode holonomy operator. For the choice of $m_i = 1$, the

²Note that for $p \equiv 3 \pmod{4}$, we have $p^* = -p$. Thus we may not properly define the Legendre symbol $\lambda_{p^*}(x)$ ($x \in \mathbf{F}_{p^*}^\times$). However, as far as the calculation of the Gauss sum (5.6) is concerned, we can properly define it with p^* . We shall discuss this point later in (5.36).

factor of $(-1)^{m_i}$ becomes $\lambda_{\alpha_i}(\beta)$. Thus the relation (4.55) can be written as

$$\begin{aligned}\langle \Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau) \rangle &= \prod_i^n \lambda_{\alpha_i}(\beta) \langle \Theta_{R,\gamma}^{(\tau)}(a) \rangle \\ &= \left(\frac{\beta}{\alpha_1 \alpha_2 \cdots \alpha_n} \right) \langle \Theta_{R,\gamma}^{(\tau)}(a) \rangle \\ &= \eta_{N_1}(\beta) \langle \Theta_{R,\gamma}^{(\tau)}(a) \rangle\end{aligned}\quad (5.16)$$

where we use the notation (4.49) and $\eta_{N_1}(\beta)$ is the Jacobi symbol, with α_i ($i = 1, 2, \dots, n$) and β denoting odd primes and an arbitrary integer, respectively. N_1 is an odd integer defined by

$$N_1 = \alpha_1 \alpha_2 \cdots \alpha_n. \quad (5.17)$$

The general form of the Jacobi symbol, in the form of (5.10), can also be obtained by relaxing the choice of winding numbers $n_i \in \mathbf{Z}$. Let elements of $\{p_1, p_2, \dots, p_s\}$ be

$$\{p_1, p_2, \dots, p_s\} \in \{\alpha_1, \alpha_2, \dots, \alpha_n\} \quad (s \leq n), \quad (5.18)$$

say, $\{p_1, p_2, \dots, p_s\} = \{\alpha'_1, \alpha'_2, \dots, \alpha'_s\}$, and let the corresponding winding numbers of the β cycle be

$$\{e_1, e_2, \dots, e_s\} \in \{n_1, n_2, \dots, n_n\}, \quad (5.19)$$

say, $\{e_1, e_2, \dots, e_s\} = \{n'_1, n'_2, \dots, n'_s\}$ ($s \leq n$), with the rest of n_i 's being zero. In this case, non-vanishing “phase factors” in (4.55) are given by

$$(-1)^{lk(\alpha'_j, \beta^{n'_j})} = \lambda_{\alpha'_j}(\beta^{n'_j}) = \left(\frac{\beta^{n'_j}}{\alpha'_j} \right) = \left(\frac{\beta}{\alpha'_j} \right)^{n'_j} \quad (5.20)$$

($j = 1, 2, \dots, s \leq n$) where we use the multiplicative property of the Legendre symbols

$$\left(\frac{x_1 x_2}{p} \right) = \left(\frac{x_1}{p} \right) \left(\frac{x_2}{p} \right). \quad (5.21)$$

Then the general form of the Jacobi symbol arises from the vacuum expectation value of the zero-mode holonomy operator as follows:

$$\begin{aligned}\langle \Theta_{R,\gamma}^{(\tau)}(a+1+\sqrt{-1}n'\text{Im}\tau) \rangle &= \lambda_{\alpha'_1}(\beta)^{n'_1} \lambda_{\alpha'_2}(\beta)^{n'_2} \cdots \lambda_{\alpha'_s}(\beta)^{n'_s} \langle \Theta_{R,\gamma}^{(\tau)}(a) \rangle \\ &= \eta_N(\beta) \langle \Theta_{R,\gamma}^{(\tau)}(a) \rangle\end{aligned}\quad (5.22)$$

where we use (5.10). The argument of $\Theta_{R,\gamma}^{(\tau)}(a+1+\sqrt{-1}n'\text{Im}\tau)$ means that $m_i = 1$ and $n'_j = 1$, with the rest of n_i 's being zero. N is an arbitrary odd integer represented by the prime factorization

$$N = \alpha_1'^{n'_1} \alpha_2'^{n'_2} \cdots \alpha_s'^{n'_s} = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}. \quad (5.23)$$

This shows that, even at classical level, we can extract information on prime factorization of integers (or irreducible representation of integers) in terms of the zero-mode holonomy operator applied to the knot-prime correspondence. Thus the result (5.22) illustrates a classical realization of the concept in (1.5). In what follows, we shall consider quantum aspects of this concept. It will turn out that the quantum aspects are something more profound than the classical ones discussed above.

Quantum realization of linking numbers

From the relation (3.31), we find that the operator ∂_{a_i} in $\Theta_{R,\gamma}^{(\tau)}(a)$ can be replaced by $\frac{\pi}{\text{Im}\tau} \bar{a}_i$ in the \bar{a}_i -representations. Under the transformation of $a_i \rightarrow a_i + m_i + i\text{Im}\tau$, this factor changes as

$$\frac{\pi}{\text{Im}\tau} \bar{a}_i \longrightarrow \frac{\pi}{\text{Im}\tau} (\bar{a}_i + m_i - i\text{Im}\tau) = \frac{\pi}{\text{Im}\tau} \bar{a}_i - i\pi \left(1 + i\frac{m_i}{\text{Im}\tau}\right). \quad (5.24)$$

Substituting this relation into (4.55), we find

$$(-1)^{m_i} \leftrightarrow \left\langle e^{-i\pi(1+i\frac{m_i}{\text{Im}\tau})} \right\rangle. \quad (5.25)$$

By the choice of $m_i = 1$ as in (5.14), this relation reduces to

$$(-1)^{m_i}|_{m_i=1} = (-1)^{lk(\alpha_i, \beta)} \leftrightarrow \left\langle (-1)^{1+i\frac{1}{\text{Im}\tau}} \right\rangle. \quad (5.26)$$

Since $\text{Im}\tau$ is simply a real parameter, the vacuum expectation value in (5.26) does not make sense unless we consider the factor (-1) as some operator. Naturally, this operator can be identified with the Legendre symbol. This will be obvious from the correspondence between knots and primes discussed in (5.14) and (5.15). As mentioned below (5.15), there are essentially two interpretations to the Legendre symbols in connection with the linking number. Here we shall consider the following two cases:

$$(-1) = \begin{cases} (-1)^{lk(\alpha_i, \beta)} = \lambda_{\alpha_i}(\beta); \\ (-1)^{lk(\beta^*, \alpha_i)} = \lambda_{\beta^*}(\alpha_i). \end{cases} \quad (5.27)$$

The first interpretation is to consider α_i as an odd prime number and β as an integer ($x \in \mathbf{F}_{\alpha_i}^\times$), while the second one is to consider β as an odd prime and α_i as an integer ($x \in \mathbf{F}_{\beta^*}^\times$). The first case involves multiple prime numbers for $i = 1, 2, \dots, n$ and this will lead to give a quantum or operative version of the relation in (5.16). On the other hand, the second case involves a single prime number and this case will be more convenient to deal with “dynamics” of primes in the zero-mode holonomy formalism, particularly in relation with Riemann’s zeta function. In the following, we shall study details of these considerations.

Analogs of scattering amplitudes for prime numbers

We begin with the first case of (5.27). From (4.55) and (5.26), we can express the “gauged” abelian holonomy operator $\Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau)$ as

$$\begin{aligned}
& \Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau; \alpha_i, \beta) \\
&= \text{Tr}_{R,\gamma} \exp \left[\sum_{r \geq 2} (\lambda_{\alpha_1}(\beta) \lambda_{\alpha_2}(\beta) \cdots \lambda_{\alpha_r}(\beta))^{1+\frac{i}{\text{Im}\tau}} \right. \\
& \quad \left. \partial_{a_1} \otimes \partial_{a_2} \otimes \cdots \otimes \partial_{a_r} (-1)^r \zeta(r) \right] \\
&= \exp \left[\sum_{r \geq 2} \sum_{\sigma \in \mathcal{S}_{r-1}} \eta_{N_1}(\beta)^{1+\frac{i}{\text{Im}\tau}} \partial_{a_1} \otimes \partial_{a_{\sigma_2}} \otimes \partial_{a_{\sigma_3}} \otimes \cdots \otimes \partial_{a_{\sigma_r}} (-1)^r \zeta(r) \right].
\end{aligned} \tag{5.28}$$

Here N_1 is given by $N_1 = \alpha_1 \alpha_2 \cdots \alpha_r$, with α_i ($i = 1, 2, \dots, r$) denoting distinct odd prime numbers. An analog of a scattering amplitude of these primes, *i.e.*, the irreducible constituents of the odd integer N_1 , is then written as

$$\begin{aligned}
& \mathcal{A}(N_1, \beta) \Big|_{N_1=\alpha_1 \alpha_2 \cdots \alpha_n} \\
&\equiv \frac{1}{(n-1)!} \frac{\delta}{\delta \partial_{a_1}} \otimes \frac{\delta}{\delta \partial_{a_2}} \otimes \cdots \otimes \frac{\delta}{\delta \partial_{a_n}} \Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau; \alpha_i, \beta) \Big|_{\partial_a=0} \\
&= \eta_{N_1}(\beta)^{1+\frac{i}{\text{Im}\tau}} (-1)^n \zeta(n)
\end{aligned} \tag{5.29}$$

where $\partial_a = 0$ in the second line means that the remaining ∂_{a_i} operators (or source functions to be precise) all vanish upon the completion of the functional derivatives. Notice that this expression is in the same form as the scattering amplitudes of photons in the holonomy formalism except that we

do not use physical operators a_i^\pm but the zero-mode operators ∂_{a_i} here. (For the description of holonomy operators as S-matrix functionals of scattering amplitudes for gluons in general, see [3].) Thus we can naturally interpret $\eta_{N_1}(\beta)^{1+\frac{i}{\text{Im}\tau}}\zeta(n)$ as an analog of scattering amplitude for n prime numbers, with their values $(\alpha_1, \alpha_2, \dots, \alpha_n)$ not being specified. In this sense, the zero-mode holonomy operator gives a generating function for the “scattering amplitudes” of prime numbers. As mentioned in the introduction, obtaining the expression (5.29) is a main goal of the present paper.

Any physical observable is given by the square of a probability amplitude. The “scattering probability” corresponding to $\mathcal{A}(N_1, \beta)$ is then expressed, up to normalization, as

$$|\mathcal{A}(N_1, \beta)|^2 = \zeta(n)^2 \quad (5.30)$$

where we use the fact that the value of the Jacobi symbol $\eta_{N_1}(\beta)$ is nothing but ± 1 ; note that each of the Legendre symbols $\lambda_{\alpha_i}(\beta)$ takes a value of ± 1 .

The expressions (5.29) and (5.30) are independent of the choices of n prime numbers $(\alpha_1, \alpha_2, \dots, \alpha_n)$. In fact, we can calculate these quantities even with $\alpha_i = 1$ ($i = 1, 2, \dots, n$):

$$\begin{aligned} & \mathcal{A}(N_0, \beta)|_{N_0=\underbrace{1 \times 1 \times \dots \times 1}_n} \\ \equiv & \frac{1}{(n-1)!} \frac{\delta}{\delta \partial_{a_1}} \otimes \frac{\delta}{\delta \partial_{a_2}} \otimes \dots \otimes \frac{\delta}{\delta \partial_{a_n}} \Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau; \alpha_i, \beta) \Big|_{\partial_a=0} \\ = & \eta_{N_0}(\beta)^{1+\frac{i}{\text{Im}\tau}} (-1)^n \zeta(n) = (-1)^n \zeta(n) \end{aligned} \quad (5.31)$$

where we use the conventional definition of the Jacobi symbol

$$\eta_1(\beta) = \left(\frac{\beta}{1} \right) = 1. \quad (5.32)$$

Notice that the result (5.31) can also be obtained by use of $\Theta_{R,\gamma}^{(\tau)}(a)$ in (4.51), the “pure-gauge” holonomy operator. This confirms that we can properly understand $\Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau; \alpha_i, \beta)$ as a gauged operator of $\Theta_{R,\gamma}^{(\tau)}(a; \alpha_i, \beta)$.

As in the classical case, a generalization of (5.29) can be carried out by relaxing the condition for the winding numbers n_i . This leads to the expression

$$\mathcal{A}(N, \beta)|_{N=p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}}$$

$$\begin{aligned}
&\equiv \frac{1}{(n-1)!} \frac{\delta}{\delta \partial_{a_1}} \otimes \frac{\delta}{\delta \partial_{a_2}} \otimes \cdots \otimes \frac{\delta}{\delta \partial_{a_n}} \Theta_{R,\gamma}^{(\tau)}(a+1 + \sqrt{-1}n' \text{Im}\tau; \alpha_i, \beta) \Big|_{\partial_a=0} \\
&= \eta_N(\beta)^{1+\frac{i}{\text{Im}\tau}} (-1)^n \zeta(n)
\end{aligned} \tag{5.33}$$

where we follow the same notations in (5.18)-(5.23). The resultant “scattering amplitude” is essentially same as (5.29) except that now the prime factorization of the integer N is given by (5.23) rather than that of N_1 in (5.17). The squares of these amplitudes are all identical to $\zeta(n)^2$, *i.e.*,

$$|\mathcal{A}(N_0, \beta)|^2 = |\mathcal{A}(N_1, \beta)|^2 = |\mathcal{A}(N, \beta)|^2 = \zeta(n)^2. \tag{5.34}$$

This means that the normalized scattering probability is always one. Physically this is obvious because once we choose N (and β) to calculate the amplitudes, the irreducible representation of N , *i.e.*, a set of the values of α_i , n_i (and $m_i = 1$), is determined by hand so that there is no notion of probability in the existence of N . In other words, once an odd integer which factorizes into odd primes is given, there arises no notion of creation for the integer; it is already there. Thus it is probably not appropriate to calculate quantities such as scattering probabilities of primes, which appear in a particular prime factorization, in a framework of quantum field theory.

One may consider that the above argument can be improved by integrating out the β -dependence. This means the use of the Gauss sum for each of $\lambda_{\alpha_i}(\beta)$ in (5.28). In calculating the Gauss sum, we need to sum over $\beta \in \mathbf{F}_{\alpha_i}^\times$ for each of the primes α_i . However, since there are n distinct such primes, it is impossible to execute a set of summations with a single parameter β ; we need multiple β 's corresponding to distinct α_i 's. Thus the β -dependence can not be integrated out in the present interpretation of α_i and β . This becomes possible if we choose the other interpretation of α_i and β , namely, the second interpretation of (5.27). We shall consider this case in the following.

The Gauss sum as a prime-creation operator

Applying the second interpretation of (5.27) to the expression (5.28), we can similarly write down the gauged abelian holonomy operator $\Theta_{R,\gamma}^{(\tau)}(a+1 + i\text{Im}\tau)$ as

$$\begin{aligned}
&\Theta_{R,\gamma}^{(\tau)}(a+1 + i\text{Im}\tau; \beta, \alpha_i) \\
&= \text{Tr}_{R,\gamma} \exp \left[\sum_{r \geq 2} (\lambda_{\beta^*}(\alpha_1) \lambda_{\beta^*}(\alpha_2) \cdots \lambda_{\beta^*}(\alpha_r))^{1+\frac{i}{\text{Im}\tau}} \right]
\end{aligned}$$

$$\begin{aligned}
& \partial_{a_1} \otimes \partial_{a_2} \otimes \cdots \otimes \partial_{a_r} (-1)^r \zeta(r) \Big] \\
= & \exp \left[\sum_{r \geq 2} \sum_{\sigma \in \mathcal{S}_{r-1}} (\lambda_{\beta^*}(\alpha_1) \lambda_{\beta^*}(\alpha_2) \cdots \lambda_{\beta^*}(\alpha_r))^{1 + \frac{i}{\text{Im}\tau}} \right. \\
& \left. \partial_{a_1} \otimes \partial_{a_{\sigma_2}} \otimes \partial_{a_{\sigma_3}} \otimes \cdots \otimes \partial_{a_{\sigma_r}} (-1)^r \zeta(r) \right] \quad (5.35)
\end{aligned}$$

where β is now an odd prime and α_i ($i = 1, 2, \dots, r$) is an integer defined in a space of $\mathbf{F}_{\beta^*}^\times$. Thus, in this case, we can suitably consider a Fourier transform of $\lambda_{\beta^*}(\alpha_i)$ for each i . This means that we can replace each of $\lambda_{\beta^*}(\alpha_i)$ by

$$\hat{\lambda}_{\beta^*} = \sum_{\alpha_i \in \mathbf{F}_{\beta^*}^\times} \lambda_{\beta^*}(\alpha_i) e^{i \frac{2\pi}{\beta^*} \alpha_i} = \sqrt{\beta} \quad (5.36)$$

where we naively use the relations in (5.5) and (5.6), that is, although the Legendre symbol $\lambda_{\beta^*}(\alpha_i)$ ($\alpha_i \in \mathbf{F}_{\beta^*}^\times$) is not appropriately defined for $\beta \equiv 3 \pmod{4}$, the result $\hat{\lambda}_{\beta^*} = \sqrt{\beta}$ holds for any odd prime β by a direct application of (5.6). In fact, a naive calculation of $\hat{\lambda}_{3^*}$ suggests that $\lambda_{3^*}(x)$ ($\alpha_i \in \mathbf{F}_{3^*}^\times$) should be defined by $\sqrt{-1}$. Thus, practically speaking, it is not possible to define $\lambda_{\beta^*}(\alpha_i)$ in compatible with the definition of the Legendre symbol (1.2). However, as far as the calculation of the Gauss sum is concerned, we can bypass these issues and directly use the result in (5.36).

Using (5.36), we can properly integrate out the α_i -dependence in (5.35) to obtain

$$\begin{aligned}
& \Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau; \beta) \\
= & \text{Tr}_{R,\gamma} \exp \left[\sum_{r \geq 2} \sqrt{\beta}^{(1 + \frac{i}{\text{Im}\tau})r} \partial_{a_1} \otimes \partial_{a_2} \otimes \cdots \otimes \partial_{a_r} (-1)^r \zeta(r) \right] \quad (5.37)
\end{aligned}$$

where we express the braid trace by $\text{Tr}_{R,\gamma}$ for simplicity. As mentioned earlier, below (5.7) and also in the introduction, the Gauss sum (5.36) can be interpreted as an operator in a space of $\frac{2\pi}{\beta^*}$. It is a conjugate operator of the Legendre symbol $\lambda_{\beta^*}(\alpha_i)$ which can, on the other hand, be considered as an operator in a space of $\alpha_i \in \mathbf{F}_{\beta^*}^\times$. Notice that, in a language of quantum field theory, the Fourier transform (5.36) shows that the relevant phase space for these operators is given by $(\frac{2\pi}{\beta^*}, \alpha_i)$. *In the present framework, the Gauss sum $\hat{\lambda}_{\beta^*}$ powered by $(1 + \frac{i}{\text{Im}\tau})$ is the only operator that involves the odd*

prime number β . Thus it is natural to interpret it as a creation operator of the prime β . Notice that, as discussed in (5.26), the factor of $\left(1 + \frac{i}{\text{Im}\tau}\right)$ is necessary in taking account of quantum realization of the Legendre symbols. This point becomes crucial in the following arguments.

Now it is well-known that Riemann's zeta function $\zeta(r)$ can be expressed as a product sum over primes

$$\zeta(r) = \prod_P \frac{1}{1 - P^{-r}} = \frac{1}{1 - 2^{-r}} \prod_p \frac{1}{1 - p^{-r}}. \quad (5.38)$$

This is known as Euler's product. Here P runs over all primes including $P = 2$, while p denotes odd primes as before. The above analysis suggests that quantum theoretically the odd primes p can be replaced by the operator $(\hat{\lambda}_{p^*})^{1 + \frac{i}{\text{Im}\tau}}$, i.e.,

$$p \leftrightarrow (\hat{\lambda}_{p^*})^{1 + \frac{i}{\text{Im}\tau}} = (\sqrt{p})^{1 + \frac{i}{\text{Im}\tau}}. \quad (5.39)$$

For $P = 2$, either of the quantity P^* or the Gauss sum $\hat{\lambda}_2$ is not defined. However, one may define the Legendre symbol $\lambda_2(x)$ where $x \in \mathbf{F}_2^\times$ takes a value of $x = 1$ or -1 . In fact, it is possible to generalize the definition of the Gauss sum such that it includes the case of $\lambda_2(x)$. This is a Gauss sum that is expanded by the so-called Dirichlet character $\chi(x)$ [24]:

$$\begin{aligned} \hat{\lambda}_P &= \sum_{x \in \mathbf{F}_P} \chi(x) e^{i \frac{2\pi}{P} x} \\ &= \begin{cases} \sqrt{P} & \text{for } \chi(-1) = 1; \\ i\sqrt{P} & \text{for } \chi(-1) = -1. \end{cases} \end{aligned} \quad (5.40)$$

This can be seen as a generalization of (5.6). In this generalization, the Legendre symbol $\lambda_P(x)$ ($x \in \mathbf{F}_P$) is interpreted as the Dirichlet character $\chi(x)$. For $P = 2$, we have $\chi(-1) = \lambda_2(-1) = \left(\frac{-1}{2}\right) = 1$. Thus the Gauss sum $\hat{\lambda}_2$ can be calculated as

$$\hat{\lambda}_2 = \sqrt{2}. \quad (5.41)$$

Now we would like to define 2^* . A naive application of the definition $q^* = (-1)^{\frac{q-1}{2}} q$ leads to $2^* = i2$. However, we may consider that this is inappropriate because 2^* should take a value of either $+2$ or -2 as the rest of prime numbers. We then *define* 2^* as $2^* = 2$ so that we also obtain

$$\hat{\lambda}_{2^*} = \sqrt{2}. \quad (5.42)$$

This is a natural choice if we think of the fact that the elements of \mathbf{F}_2 and \mathbf{F}_{-2} are identical; for both cases the elements can be given by either of ± 1 . Of course, this fact itself does not justify the choice of $2^* = 2$ but there is nothing that prohibits this choice as well since the notion of 2^* is introduced in a way that is somewhat independent of the correspondence between knots and primes in (5.1). In fact, the essential quantities in the following discussions are the moduli (or the absolute values) of the Gauss sums. Thus, although the distinction between P and P^* is important in relation between linking numbers and Legendre symbols, it does not affect the modulus of the Gauss sum, $|\hat{\lambda}_P| = |\hat{\lambda}_{P^*}| = \sqrt{P}$. This indicates another justification of the choice of $2^* = 2$. With (5.42) understood along these lines of reasonings, we find that the relation (5.39) holds for any prime numbers P in general. Therefore the operator $\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau; \beta)$ in (5.37) can further be written as

$$\begin{aligned} & \Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau; \beta) \\ = & \text{Tr}_{R,\gamma} \exp \left[\sum_{r \geq 2} \beta^{\frac{r}{2}(1 + \frac{i}{\text{Im}\tau})} \partial_{a_1} \otimes \partial_{a_2} \otimes \cdots \otimes \partial_{a_r} \right. \\ & \left. (-1)^r \prod_P \frac{1}{1 - P^{-\frac{r}{2}(1 + \frac{i}{\text{Im}\tau})}} \right] \end{aligned} \quad (5.43)$$

where β is an odd prime number and P runs over all prime numbers. By use of this holonomy operator, we can calculate an analog of “scattering amplitude” for n β ’s:

$$\begin{aligned} \mathcal{A}(\beta^n) &= \frac{1}{(n-1)!} \frac{\delta}{\delta \partial_{a_1}} \otimes \frac{\delta}{\delta \partial_{a_2}} \otimes \cdots \otimes \frac{\delta}{\delta \partial_{a_n}} \Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau; \beta) \Big|_{\partial_a=0} \\ &= (-1)^n \beta^{\frac{n}{2}(1 + \frac{i}{\text{Im}\tau})} \prod_P \frac{1}{1 - P^{-\frac{n}{2}(1 + \frac{i}{\text{Im}\tau})}} \\ &= (-1)^n \beta^{\frac{n}{2}(1 + \frac{i}{\text{Im}\tau})} \zeta \left(\frac{n}{2} + i \frac{n}{2\text{Im}\tau} \right) \end{aligned} \quad (5.44)$$

where $n \geq 2$. In the holonomy formalism this can be considered as a scattering amplitude of n odd primes of the same value β . From a perspective of particle physics, we can regard β as a boson of a same species. Thus (5.44) can also be interpreted as a nobel scattering amplitude of bosons. The square of the amplitude is given by

$$|\mathcal{A}(\beta^n)|^2 = \beta^n \left| \zeta \left(\frac{n}{2} + i \frac{n}{2\text{Im}\tau} \right) \right|^2. \quad (5.45)$$

The appearance of Riemann's zeta function indicates that there is a nontrivial self-interactions among β 's depending on the number of β 's that involves in the scattering processes. In the large n limit, we have

$$\left| \zeta \left(\frac{n}{2} + i \frac{n}{2\text{Im}\tau} \right) \right|^2 \rightarrow 1 \quad (n \gg 1). \quad (5.46)$$

This suggests that the self-interaction effect vanishes for sufficiently large n . This asymptotic behavior also supports the idea that the appearance of Riemann's zeta function arises from a quantum effect of the scattering processes among identical odd primes.

Physical interpretation of the Riemann hypothesis

Riemann's zeta function $\zeta(s)$ ($s \in \mathbf{C}$) vanishes at the negative even integers, *i.e.*, $s = -2, -4, -6, \dots$. These are called the trivial zeros of $\zeta(s)$. There is a famous conjecture about nontrivial zeros of $\zeta(s)$, that is, the real part of any nontrivial zeros of $\zeta(s)$ is equal to $\frac{1}{2}$. This is called the Riemann hypothesis. In the following, we propose a physical interpretation of this hypothesis by extrapolating the result (5.44) to the case of $n = 1$.

The zero-mode holonomy operator (5.43) and the abelian holonomy operator (2.16) in general are defined for $n \geq 2$. As discussed below (2.17), we can properly define the exponent of the holonomy operators to be zero for $r = 1$. Following the expressions in (4.41)-(4.43), let us denote this quantity as

$$P^{(\tau)} \oint_{\gamma} \tilde{A} := \partial_{a_1} \oint_{\gamma} \omega^{(1)}. \quad (5.47)$$

Notice that the iterated-integral representation of Riemann's zeta function (4.41) is not defined for $r = 1$. This problem can however be solved by taking the limit of $x \rightarrow 1$ in the following relation:

$$\begin{aligned} \text{Li}_1(x) &= \int_0^x \omega^{(1)} = \int_0^x \frac{dt}{1-t} = -\log(1-x) \\ &\rightarrow \oint_{\gamma} \omega^{(1)} = \zeta(1) \quad (x \rightarrow 1) \end{aligned} \quad (5.48)$$

where γ is defined in (4.22) and $\omega^{(1)}$ is defined in (4.33), respectively. Thus, by use of the polylogarithm function, we can properly extrapolate the expression (4.41) to the case of $r = 1$. As is well-known, the resultant value $\zeta(1)$ diverges. This seems to cause a problem in defining the quantity (5.47)

being zero. But this can easily be remedied by assuming that the expectation value of ∂_{a_1} or $\frac{\pi}{\text{Im}\tau}\bar{a}_1$ can be expanded by factors of $|1-x|^\epsilon$ with $\epsilon > 0$. In principle, such an assumption can always be imposed as there are no restrictions on the asymptotic behaviors of the zero-mode variables at $x \rightarrow 1$.

The above argument shows that we can properly define the exponent of the zero-mode holonomy operator even for $r = 1$. The gauged holonomy operator (5.43) is obtained from the application of the zero-mode holonomy operator (4.51) to a space of odd prime number β . The structure of the gauged holonomy operator (5.43) and the origin of Riemann's zeta function in particular remain the same as the original holonomy operator (4.51). The difference arises from the introduction of the Gauss-sum operator (powered by a certain factor) which we interpret as a creation operator for the odd prime β . We then further rewrite the zeta function in terms of this prime-creation operator, using the formula of Euler's product. Therefore we can similarly define the case of $r = 1$ for the gauged holonomy operator (5.43) as well. This means that the “scattering amplitude” in (5.44) can be applied for $n = 1$, giving an expression

$$\mathcal{A}(\beta) = -\beta^{\left(\frac{1}{2} + i\frac{1}{2\text{Im}\tau}\right)} \zeta\left(\frac{1}{2} + i\frac{1}{2\text{Im}\tau}\right) = 0. \quad (5.49)$$

This is nothing but the indication of the Riemann hypothesis in the abelian holonomy formalism.

Physically it is obvious that the amplitude (5.49) vanishes because there is no notion of “scattering” for a single-particle system. In other words, there is no notion of “interaction” in a one-body system. This assertion is more likely true for a system of a massless boson. In this sense, the result (5.49) can be physically well-understood. Furthermore, in the present case, we no longer deal with the divergent $\zeta(1)$ so that here we do not have to really worry about the asymptotic behavior of the zero-mode variables which we have discussed above.

The imaginary part of the nontrivial zeros of $\zeta(s)$ are numerically given by 14.13, 21.02, 25.01, \dots for $\text{Im}s > 0$. The largest value of $\text{Im}\tau$ that satisfies (5.49) is then evaluated as

$$(\text{Im}\tau)_{\text{max}} = 0.03537. \quad (5.50)$$

This value decreases towards zero as $\text{Im}s$ increases. Since $\tau = i\text{Im}\tau$ is the modular parameter of the torus, the value of $\text{Im}\tau$ characterizes the shape of

the torus, *i.e.*, the Riemann surface on which we define the two-dimensional conformal field theory.

6 Concluding remarks

In the present paper, we further investigate an abelian version of the holonomy formalism that is recently developed as a nonperturbative approach to non-abelian gauge theories [3]. We first review the construction of a holonomy operator in two-dimensional conformal field theory and then consider zero-mode contributions to it. The zero-mode analysis is made possible by use of a geometric-quantization scheme. We then develop a method to extract a purely zero-mode part of the abelian holonomy operator by redefining the “path ordering” in the operator. It turns out that the zero-mode holonomy operator $\Theta_{R,\gamma}^{(\tau)}(a)$ can be expressed in terms of Riemann’s zeta function.

Along the way, we also show that a generalization of linking numbers can be obtained in terms of the vacuum expectation values of the zero-mode holonomy operators. This is shown by use of an abelian gauge theory on the zero-mode variables $a_i \in \mathbf{C}$ ($i = 1, 2, \dots, n$) where gauge transformations are realized by the double periodicity condition $a_i \rightarrow a_i + m_i + n_i\tau$ ($m_i, n_i \in \mathbf{Z}$). This means that a_i is nothing but a complex coordinate on a torus with a modular parameter τ which we set to $\tau = i\text{Im}\tau$. The integers m_i and n_i correspond to the winding numbers of α_i and β_i cycles for the i -th torus, respectively. As discussed in the end of chapter 3, by fixing one of these numbers, we can interpret the other as a linking number between the α_i and β_i cycles.

Inspired by mathematical analogies between linking numbers and Legendre symbols, we then consider an application of the gauged zero-mode holonomy operator $\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau)$ to a space of \mathbf{F}_p^\times where p is an odd prime number. This enables us to incorporate Legendre symbols into $\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau)$ such that they can be treated as operators in the space of \mathbf{F}_p^\times . We make both classical and quantum analyses on this new holonomy operator and show that it naturally leads to the so-called Jacobi symbols by a suitable choice of α_i and β_i , with β_i being identical for any i ’s.

Furthermore, we utilize the fact that the Gauss sum can be interpreted as a Fourier transform (or a Fourier expansion, to be more precise) of the Legendre symbol. An analog of the phase space that is relevant to the Fourier

transform is given by $\left(\frac{2\pi}{p}, x\right)$ with $x \in \mathbf{F}_p^\times$. Thus, in the conjugate space of $\frac{2\pi}{p}$, the Legendre symbol can be represented by the Gauss sum. We apply these mathematical facts to $\Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau)$ and construct a gauged zero-mode holonomy operator $\Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau;\beta)$ in a space that is conjugate to a space of $\mathbf{F}_{\beta^*}^\times$, with β^* defined by $\beta^* = (-1)^{\frac{\beta-1}{2}}\beta$. An explicit form of the operator $\Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau;\beta)$ is shown in (5.43).

What is interesting, if not mathematically rigorous, in obtaining the expression (5.43) is that we identify the Gauss-sum operator $\hat{\lambda}_{\beta^*}$ as a creation operator of an odd prime β . As mentioned in section 5, in an operator language, the Gauss sum powered by the factor of $\left(1 + \frac{i}{\text{Im}\tau}\right)$ is the only operator that involves the odd prime β . Thus it is natural, at least from a perspective of physicists', to interpret $\hat{\lambda}_{\beta^*}$ as an operator that is relevant to the creation of prime number β . An explicit correspondence is given in (5.39). In section 5, we also discuss that this interpretation can naturally be extended to the case of the even prime number, $\beta = 2$. By use of this correspondence and the formula of Euler's product, we can therefore make a persuasive argument to express Riemann's zeta function in terms of the Gauss-sum operators. One may understand this expression as a consequence of the application of abelian holonomy formalism to the elementary theory of numbers. In this sense, the expression $\Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau;\beta)$ in (5.43) presents a quantum realization of Riemann's zeta function in the holonomy formalism.

The holonomy formalism is a kind of universal formalism that uses a holonomy operator of conformal field theory as an S-matrix functional for scattering amplitudes of massless bosons. In this context, we can identify the gauged zero-mode holonomy operator $\Theta_{R,\gamma}^{(\tau)}(a+1+i\text{Im}\tau;\beta)$ as a generating function for an analog of "scattering amplitude" for identical prime numbers. The resultant "scattering amplitude" is then computed in (5.44). We argue that this result can also be applied to the case of a single-particle, or a single-prime, system and shows that in this case the result (5.44) provides a novel indication of the Riemann hypothesis. Physically this result arises from an obvious fact that there is no notion of interaction in a single-particle system.

Lastly, we would like to remark that our construction of zero-mode holonomy operators provides an interesting framework for studies on quantum aspects of topology and number theory. The gauged zero-mode holonomy operators in forms of (5.28) and (5.43) define abelian gauge theories on zero

modes. These are simple $U(1)$ gauge theories but, as we have seen explicitly, corresponding physical results (5.34) and (5.45) hold regardless the choices of α_i and β which correspond to either an odd prime number or an integer. The choices depend on how we realize the correspondence between linking numbers and Legendre symbols. Furthermore, as shown in (5.33), for a certain choice we do not have to fix the winding numbers m_i and n_i , or at least one of them, to the identity. Thus the abelian gauge theories which we construct have indeed very rich properties in terms of topology and number theory, and will be useful for studies of these subjects in a framework of quantum field theory.

Appendix: Index of notations

$a_i \ (\rightarrow a_i + m_i + n_i \tau)$	complex variables for zero modes ($m_i, n_i \in \mathbf{Z}$)
$a_i^{(\pm)}$	creation operators of photons with helicity \pm
$A = \frac{1}{\kappa} \sum_{i < j} A_{ij} \omega_{ij}$	corresponding to the ladder operators of $SL(2, \mathbf{C})$
$A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}$	comprehensive gauge fields for photons
$A^{(\tau)} = \sum_{i < j} A_{ij}^{(\tau)} \omega_{ij}$	bialgebraic operators
$A_{ij}^{(\tau)} = 1_i \otimes \partial_{a_j}$	comprehensive gauge fields for zero modes
$\mathcal{B}_n = \Pi_1(\mathcal{C})$	bialgebraic operators for zero modes
$\mathcal{C} = \mathbf{C}^n / \mathcal{S}_n$	braid group
$\eta_N(y) = (\frac{y}{N})$	physical configuration space
$\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$	Jacobi symbol ($y \in \mathbf{Z}$)
$\mathbf{F}_p^\times = \mathbf{F}_p - \{0\}$	finite field (reduced residue class group)
$i, j \ (= 1, 2, \dots, n)$	space of mod p
$k \ (= 1)$	numbering index
κ	level number of the $U(1)$ Chern-Simons theory
$K(a_i, \bar{a}_i)$	KZ parameter
$\lambda_p(q) = \left(\frac{q}{p}\right)$	Kähler potential for zero modes
$\hat{\lambda}_p = \sum_{x \in \mathbf{F}_p^\times} \lambda_p(x) \exp\left(\frac{i2\pi}{p}x\right)$	Legendre symbol
$m_i, n_i \ (= 1)$	Gauss sum, equal to $\sqrt{p^*}$
$N = p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$	winding numbers of α_i and β_i cycles, respectively
$\omega_{ij} = d \log(z_i - z_j)$	prime factorization of odd positive integer N
$\omega_j^{(0)} = \frac{dz_j}{z_j}$	logarithmic one-form
	logarithmic one-form (type 0)

$\omega_j^{(1)} = \frac{-dz_j}{1-z_j}$	logarithmic one-form (type 1)
$\Omega = \frac{1}{\kappa} \sum_{i < j} \Omega_{ij} \omega_{ij}$	KZ connection
$\Omega_{ij} = \sum_{\mu} a_i^{(\mu)} \otimes a_j^{(\mu)}$	bialgebraic operators, $\mu (= 0, 1, 2)$ corresponding to the full indices for the elements of $SL(2, \mathbf{C})$
$\Omega^{(\tau)}$	zero-mode Kähler form on torus
p, q	odd prime numbers
$p^* = (-1)^{\frac{p-1}{2}} p$	odd primes with sign \pm
P	prime numbers (including $P = 2$)
\mathcal{S}_n	rank- n symmetric group
$\tau = \text{Re}\tau + i\text{Im}\tau = i\text{Im}\tau$	modular parameter of torus, setting $\text{Re}\tau = 0$
$\Theta_{R,\gamma}(z)$	abelian holonomy operator of A
$\Theta_{R,\gamma}(z, a)$	abelian holonomy operator of $\tilde{A} = A + A^{(\tau)}$
$\Theta_{R,\gamma}^{(\tau)}(a)$	zero-mode holonomy operator; $A^{(\tau)}$ -part of $\Theta_{R,\gamma}(z, a)$
$\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau)$	Gauged zero-mode holonomy operator
$\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau; \beta, \alpha_i)$	$\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau)$ incorporated with $\lambda_{\beta^*}(\alpha_i)$ where β is an odd prime and $\alpha_i \in \mathbf{F}_{\beta^*}^\times$
$\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau; \beta)$	$\Theta_{R,\gamma}^{(\tau)}(a + 1 + i\text{Im}\tau)$ incorporated with $\hat{\lambda}_{\beta^*}$
$V^{\otimes n} = V_1 \otimes V_2 \otimes \cdots \otimes V_n$	physical Hilbert space
V_i	Fock space for the i -th photon
$\zeta(r) = \sum_{n \geq 1} \frac{1}{n^r}$	Riemann's zeta function

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